Planar quadratic differential systems with invariant straight lines of the total multiplicity 4

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Abstract

In this article we consider the action of affine group and time rescaling on planar quadratic differential systems. We construct a system of representatives of the orbits of systems with four invariant lines, including the line at infinity and including multiplicities. For each orbit we exhibit its configuration. We characterize in terms of algebraic invariants and comitants and also geometrically, using divisors of the complex projective plane, the class of quadratic differential systems with four invariant lines. These conditions are such that no matter how a system may be presented, one can verify by using them whether the system has exactly four invariant lines including multiplicities, and if it is so, to check to which orbit (or family of orbits) it belongs.

1 Introduction

We consider here real planar differential systems of the form

$$(S)\qquad \frac{dx}{dt}=p(x,y),\qquad \frac{dy}{dt}=q(x,y), \tag{1.1}$$

where $p,\ q\in\mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$, and their associated vector fields

$$\tilde{D}=p(x,y)\frac{\partial}{\partial x}+q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

Each such system generates a complex differential vector field when the variables range over $\mathbb{C}$. To the complex systems we can apply the work of Darboux on integrability via invariant algebraic curves (cf. [5]).

For the systems (1.1) we can use the following definition.

Definition 1.1. *An affine algebraic invariant curve of a polynomial system* (1.1) *(or an algebraic particular integral) is a curve* $f(x,y)=0$ *where* $f\in\mathbb{C}[x,y]$, $\deg(f)\geq 1$, *such that there exists* $k(x,y)\in\mathbb{C}[x,y]$ *satisfying* $\tilde{D}f=fk$ *in* $\mathbb{C}[x,y]$. *We call* k *the cofactor of* f *with respect to the system.*

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We are interested in polynomial systems (1.1) possessing algebraic invariant curves. The presence of a sufficient number of such curves implies integrability of the system via the geometric method of integration of Darboux (cf. [5]). For a brief introduction to the work of Darboux we refer to the survey article [21]. Some applications of the work of Darboux in connection with the problem of the center are given in [22]. There is a growing literatures on problems related to the work of Darboux on invariant algebraic curves of differential equations. In particular we mention here the recent work of C. Christopher, J.V. Perreira and J. Llibre [4] on the notion of multiplicity of an invariant algebraic curve of a differential system.

In this article we shall consider the simplest kind of such a structure, i.e. quadratic systems (1.1) possessing invariant lines. Some references on this topic are: [2, 6, 11, 12, 18, 20, 27, 28, 31].

To a line $f(x, y) = ux + vy + w = 0$, $(u, v) \neq (0, 0)$ we associate its projective completion $F(X, Y, Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y : 1]$. The line $Z = 0$ is called the line at infinity of the system (1.1). It follows from the work of Darboux that each system of differential equations of the form (1.1) yields a differential equation on the complex projective plane which is the compactification of the complex system (1.1) on $\mathbb{P}_2(\mathbb{C})$. The line $Z = 0$ is an invariant manifold of this complex differential equation.

Notation 1.1. *Let us denote by*

$$\begin{aligned}
\mathbf{QS} &= \left\{ S \;\middle|\; \begin{array}{c} S \text{ is a system } (1.1) \text{ such that } \gcd(p(x,y), q(x,y)) = 1 \\ \text{and} \quad \max\big(\deg(p(x,y)), \deg(q(x,y))\big) = 2 \end{array} \right\};\\
\mathbf{QSL} &= \left\{ S \in \mathbf{QS} \;\middle|\; \begin{array}{c} S \text{ possesses at least one invariant affine line or} \\ \text{the line at infinity with multiplicity at least two} \end{array} \right\}.
\end{aligned}$$

For the multiplicity of the line at infinity see [24].

We shall call *degenerate quadratic differential system* a system (1.1) with $\deg \gcd(p(x, y), q(x, y)) \geq 1$ and $\max\big(\deg(p(x, y)), \deg(q(x, y))\big) = 2$.

Proposition 1.1. [2] *The maximum number of invariant lines (including the line at infinity and including multiplicities) which a quadratic system could have is six.*

Notation 1.2. *To a quadratic system* (1.1) *we can associate a point in* $\mathbb{R}^{12}$, *the ordered tuple of the coefficients of* $p(x, y)$, $q(x, y)$ *and this correspondence is an injection*

$$\begin{aligned}
\mathcal{B} : \quad & \mathbf{QS} \hookrightarrow \mathbb{R}^{12} \\
& S \;\; \mapsto \;\; \boldsymbol{a} = \mathcal{B}(S).
\end{aligned} \tag{1.3}$$

The topology of $\mathbb{R}^{12}$ *yields an induced topology on* $\mathbf{QS}$.

We associate to each system in **QSL** its *configuration* of invariant lines, i.e. the set of its invariant lines together with the singular points of the systems located on the union of these lines. In analogous manner to how we view the phase portraits of the systems on the Poincaré disc (see e.g. [10]), we can also view the configurations of real lines on the disc. To help imagining the full configurations, we complete the picture by drawing dashed lines whenever these are complex.

On the class of quadratic systems acts the group of affine transformations and time rescaling. Since quadratic systems depend on 12 parameters and since this group depends on 7 parameters, the class of quadratic systems modulo this group action, actually depends on five parameters.

It is clear that the configuration of invariant lines of a system is an affine invariant.

Definition 1.2. *We say that an invariant straight line* $\mathcal{L}(x, y) = ux + vy + w = 0$ *for a quadratic vector field* $\tilde{D}$ *has multiplicity* m *if there exists a sequence of quadratic vector fields* $\tilde{D}_k$ *converging to* $\tilde{D}$, *such that each* $\tilde{D}_k$ *has* m *distinct invariant straight lines* $\mathcal{L}_k^1 = 0, \ldots, \mathcal{L}_k^m = 0$, *converging to* $\mathcal{L} = 0$ *as* $k \to \infty$ *(with the topology of their coefficients), and this does not occur for* $m + 1$.

The notion of multiplicity thus defined is invariant under the group action, i.e. if a quadratic system (S) has an invariant line l of multiplicity m, then each system $(\tilde{S})$ in the orbit of (S) under the group action has an invariant line l of the same multiplicity m.

In this article we continue the work initiated in [24] and consider the case when the system (1.1) has exactly four invariant lines considered with their multiplicities.

The problems which we solve in this article are the following:

I) Construct a system of representatives of the orbits of systems with exactly four invariant lines, including the line at infinity and including multiplicities. For each orbit exhibit its configuration.

II) Characterize in terms of algebraic invariants and comitants and also geometrically, using divisors of the complex projective plane, the class of quadratic differential systems with four invariant lines. These conditions should be such that no matter how a system may be presented to us, we should be able to verify by using them whether the system has or does not have four invariant lines and to check to which orbit or perhaps family of orbits it belongs.

Our main results are formulated in Theorem 4.1. Theorem 4.1 gives a a complete list of representatives of the orbits of systems with exactly four invariant lines including the line at infinity and including multiplicities. These representatives are classified in 12 two-parameters families, 28 one-parameter families and 6 concrete systems. We characterize each one of these 40 families in terms of algebraic invariants or comitants and also geometrically. As the calculation of invariants and comitants can be implemented on a computer, this verification can be done by a computer.

The invariants and comitants of differential equations used in the classification Theorem 4.1 are obtained following the theory established by K.Sibirsky and his disciples (cf. [25], [26], [29], [19]).

2 Divisors associated to invariant lines configurations

Consider real quadratic systems, i.e. systems of the form:

$$(S) \qquad \begin{cases} \dfrac{dx}{dt} &= p_0 + p_1(x,y) + \ p_2(x,y) \equiv p(x,y), \\ \dfrac{dx}{dt} &= q_0 + q_1(x,y) + \ q_2(x,y) \equiv q(x,y) \end{cases} \tag{2.1}$$

with $\max(\deg(p), \deg(q)) = 2$, $\gcd(p,q) = 1$ and

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of system (2.1) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

Notation 2.1. *Let us denote by* $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$ *a point in* $\mathbb{R}^{12}$. *Each particular system* (2.1) *yields an ordered 12-tuple* $\boldsymbol{a}$ *of its coefficients.*

Notation 2.2. *Let*

$$\begin{aligned} P(X,Y,Z) =& p_0(\boldsymbol{a})Z^2 + p_1(\boldsymbol{a},X,Y)Z + \ p_2(\boldsymbol{a},X,Y) = 0, \\ Q(X,Y,Z) =& q_0(\boldsymbol{a})Z^2 + q_1(\boldsymbol{a},X,Y)Z + \ q_2(\boldsymbol{a},X,Y) = 0. \end{aligned}$$

We denote $\quad \sigma(P,Q) = \{w \in \mathbb{P}_2(\mathbb{C}) \mid P(w) = Q(w) = 0\}$.

Definition 2.1. *A formal expression of the form* $\mathbf{D} = \sum n(w)w$ *where* $w \in \mathbb{P}_2(\mathbb{C})$, $n(w)$ *is an integer and only a finite number of the numbers* $n(w)$ *are not zero, will be called a zero-cycle of* $\mathbb{P}_2(\mathbb{C})$ *and if* w *only belongs to the line* $Z = 0$ *will be called a divisor of this line. We call degree of the expression* $\mathbf{D}$ *the integer* $\deg(\mathbf{D}) = \sum n(w)$. *We call support of* $\mathbf{D}$ *the set* $\mathrm{Supp}\,(\mathbf{D})$ *of points* w *such that* $n(w) \neq 0$.

Definition 2.2. *Let* $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$.

$$\begin{aligned}
\mathbf{D}_{S}(P,Q) &= \sum_{w\in\sigma(P,Q)} I_w(P,Q)w;\\
\mathbf{D}_{S}(C,Z) &= \sum_{w\in\{Z=0\}} I_w(C,Z)wa \quad \textit{if} \quad Z \nmid C(X,Y,Z);\\
\mathbf{D}_{S}(P,Q;Z) &= \sum_{w\in\{Z=0\}} I_w(P,Q)w;\\
\widehat{\mathbf{D}}_{S}(P,Q,Z) &= \sum_{w\in\{Z=0\}} \Big(I_w(C,Z),\, I_w(P,Q)\Big)w,
\end{aligned}$$

where $I_w(F,G)$ *is the intersection number (see, [7]) of the curves defined by homogeneous polynomials* $F,\ G \in \mathbb{C}[X,Y,Z]$ *and* $\deg(F), \deg(G) \geq 1$.

Notation 2.3.

$$\begin{aligned}
n_{\mathbb{R}}^{\infty} &= \#\{w \in \mathit{Supp}\,\mathbf{D}_{S}(C,Z) \,\big|\, w \in \mathbb{P}_2(\mathbb{R})\};\\
d_{\sigma}^{\infty} &= \deg \mathbf{D}_{S}(P,Q;Z).
\end{aligned} \tag{2.2}$$

A complex projective line $uX + vY + wZ = 0$ is invariant for the system (S) if either it coincides with $Z = 0$ or is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 2.4. *Let* $S \in \mathbf{QSL}$. *Let us denote*

$$\mathbf{IL}(S) = \left\{ \; l \;\middle|\; \begin{array}{l} l \textit{ is a line in } \mathbb{P}_2(\mathbb{C}) \textit{ such} \\ \textit{that } l \textit{ is invariant for } (S) \end{array} \right\};$$
$$M(l) = \textit{the multiplicity of the invariant line } l \textit{ of } (S).$$

Remark 2.5. *We note that the line* $l_\infty : Z = 0$ *is included in* $\mathbf{IL}(S)$ *for any* $S \in \mathbf{QSL}$.

Let $l_i\ :\ f_i(x,y) = 0$, $i = 1,\ldots,k$, be all the distinct invariant affine lines (real or complex) of a system $S \in \mathbf{QSL}$. Let $l_i'\ :\ \mathcal{F}_i(X,Y,Z) = 0$ be the complex projective completion of l_i.

Notation 2.6. *We denote*

$$\mathcal{G}\ :\quad \prod_i \mathcal{F}_i(X,Y,Z)\,Z = 0; \quad \mathit{Sing}\,\mathcal{G} = \{w \in \mathcal{G} \mid\ w \textit{ is a singular point of } \mathcal{G}\};$$
$$\nu(w) = \textit{ the multiplicity of the point } w, \textit{ as a point of } \mathcal{G}.$$

Definition 2.3.

$$\begin{aligned}
\mathbf{D}_{\mathbf{IL}}(S) &= \sum_{l\in\mathbf{IL}(S)} M(l)l, \quad (S) \in \mathbf{QSL};\\
\mathit{Supp}\,\mathbf{D}_{\mathbf{IL}}(S) &= \{\, l \mid l \in \mathbf{IL}(S)\}.
\end{aligned}$$

Notation 2.7.

$$
\begin{aligned}
M_{\mathbf{IL}} &= \deg \mathbf{D}_{\mathbf{IL}}(S);\\
N_{\mathbb{C}} &= \# Supp\, \mathbf{D}_{\mathbf{IL}};\\
N_{\mathbb{R}} &= \#\{l \in Supp\, \mathbf{D}_{\mathbf{IL}} \,\big|\, l \in \mathbb{P}_2(\mathbb{R})\};\\
n^{\mathbb{R}}_{\mathcal{G},\sigma} &= \#\{\omega \in Supp\, \mathbf{D}_S(P,Q) \,|\, \omega \in \mathcal{G}\big|_{\mathbb{R}^2}\};\\
d^{\mathbb{R}}_{\mathcal{G},\sigma} &= \sum_{\omega\in\mathcal{G}|_{\mathbb{R}^2}} I_\omega(P,Q);\\
m_{\mathcal{G}} &= \max\{\nu(\omega) \,|\, \omega \in Sing\,\mathcal{G}\};\\
m^{\infty}_{\mathcal{G}} &= \max\{\nu(\omega) \,|\, \omega \in Sing\,\mathcal{G} \cap \{Z=0\}\}.
\end{aligned}
\tag{2.3}
$$

3 The main T-comitants associated to configurations of invariant lines

It is known that on the set **QS** of all quadratic differential systems (2.1) acts the group $Aff(2,\mathbb{R})$ of affine transformation on the plane (cf. [24]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2.1) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2.1) the 12-tuple $\boldsymbol{a} = (\boldsymbol{a}_{00}, \dots, \boldsymbol{b}_{02})$ of its coefficients.

For the definitions of an affine GL-comitant and invariant as well as for the definition of a T-comitant and CT-comitant we refer reader to the paper [24]. Here we shall only construct the necessary T-comitants associated to configurations of invariant lines for the class of quadratic systems with exactly four invariant lines including the line at infinity and including multiplicities.

Let us consider the polynomials

$$
\begin{aligned}
C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ \ i=0,1,2,\\
D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ \ i=1,2.
\end{aligned}
$$

As it was shown in [25] the polynomials

$$
\{\ C_0(a,x,y),\quad C_1(a,x,y),\quad C_2(a,x,y),\quad D_1(a),\quad D_2(a,x,y)\ \}
\tag{3.1}
$$

of degree one in the coefficients of systems (2.1) are GL-comitants of these systems.

Notation 3.1. *Let $f,\ g \in \mathbb{R}[a,x,y]$ and*

$$
(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\, \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.
\tag{3.2}
$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [8], [13]*)*

Theorem 3.1. [29] *Any GL-comitant of systems* (2.1) *can be constructed from the elements of the set* (3.1) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Notation 3.2. *Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a,X,Y,Z,\alpha,\beta]$ where $P = Z^2 p(X/Z, Y/Z)$, $Q = Z^2 q(X/Z,Y/Z)$, $p,\ q \in \mathbb{R}[a,x,y]$ and $\max(\deg_{(x,y)} p, \deg_{(x,y)} q) = 2$. Then*

$$
\begin{aligned}
\Phi_{\alpha,\beta} = \ & c_{11}(\alpha,\beta)X^2 + 2c_{12}(\alpha,\beta)XY + c_{22}(\alpha,\beta)Y^2 + 2c_{13}(\alpha,\beta)XZ + 2c_{23}(\alpha,\beta)YZ\\
& + c_{33}(\alpha,\beta)Z^2, \qquad \Delta(a,\alpha,\beta) = \det ||c_{ij}(\alpha,\beta)||_{i,j\in\{1,2,3\}},\\
D(a,\alpha,\beta) &= 4\Delta(a,-\beta,\alpha), \qquad H(a,\alpha,\beta) = 4\big[\det ||c_{ij}(-\beta,\alpha)||_{i,j\in\{1,2\}}\big].
\end{aligned}
$$

Lemma 3.1. [24] *Consider two parallel invariant affine lines $\mathcal{L}_i(x,y) \equiv ux + vy + w_i = 0$, $\mathcal{L}_i(x,y) \in \mathbb{C}[x,y]$, $(i = 1,2)$ of a quadratic system S of coefficients $\boldsymbol{a}$. Then $H(\boldsymbol{a},-v,u)=0$, i.e. the T-comitant $H(a,x,y)$ captures the directions of parallel invariant lines of systems* (2.1).

We construct the following T-comitants:

Notation 3.3.

$$\begin{aligned} B_3(a,x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,, \\ B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)}, \\ B_1(a) &= \operatorname{Res}_x(C_2,D)/y^9 = -2^{-9}3^{-8}(B_2,B_3)^{(4)}\,. \end{aligned} \tag{3.3}$$

Lemma 3.2. [24] *For the existence of an invariant straight line in one (respectively 2, 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$, $B_3 = 0$).*

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(a,x,y)$ and $q(a,x,y)$. We obtain $\tilde{p}(\tilde{a}(a,x_0,y_0),x',y') = p(a,x'+x_0,y'+y_0)$, $\quad \tilde{q}(\tilde{a}(a,x_0,y_0),x',y') = q(a,x'+x_0,y'+y_0)$. Let us construct the following polynomials

$$\begin{aligned} \Gamma_i(a,x_0,y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1}, \\ \Gamma_i(a,x_0,y_0) \in \mathbb{R}[a,x_0,y_0],\ (i = 1,2). \end{aligned}$$

Notation 3.4.

$$\tilde{\mathcal{E}}_i(a,x,y) = \Gamma_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i = 1,2). \tag{3.4}$$

Observation 3.5. *We note that the constructed polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$ are affine comitants of systems* (2.1) *and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{02}$ and non-homogeneous in x,y and* $\deg_a \tilde{\mathcal{E}}_1 = 3$, $\deg_{(x,y)} \tilde{\mathcal{E}}_1 = 5$, $\deg_a \tilde{\mathcal{E}}_2 = 4$, $\deg_{(x,y)} \tilde{\mathcal{E}}_2 = 6$.

Notation 3.6. *Let $\mathcal{E}_i(a,X,Y,Z)$ $(i = 1,2)$ be the homogenization of $\tilde{\mathcal{E}}_i(a,x,y)$, i.e.*

$$\mathcal{E}_1(a,X,Y,Z) = Z^5\tilde{\mathcal{E}}_1(a,X/Z,Y/Z), \qquad \mathcal{E}_2(a,X,Y,Z) = Z^6\tilde{\mathcal{E}}_1(a,X/Z,Y/Z)$$

and $\qquad \mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{E}_1(a,X,Y,Z),\ \mathcal{E}_2(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [24]):

Lemma 3.3. *The straight line $\mathcal{L}(x,y) \equiv ux + vy + w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a quadratic system* (2.1) *if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\boldsymbol{a},x,y)$ and $\tilde{\mathcal{E}}_2(\boldsymbol{a},x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y) = (ux + vy + w)\widetilde{W}_i(x,y) \quad (i = 1,2),$$

where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

Lemma 3.4. *If $\mathcal{L}(x,y) \equiv ux + vy + w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant straight line of multiplicity k for a quadratic system* (2.1) *then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{E}}_1,\tilde{\mathcal{E}}_2)$ in $\mathbb{R}[x,y]$, i.e. there exist $W_i(\boldsymbol{a},x,y) \in \mathbb{C}[x,y]$ $(i = 1,2)$ such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y) = (ux + vy + w)^k W_i(\boldsymbol{a},x,y), \quad i = 1,2. \tag{3.5}$$

Corrolary 3.7. *If the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ then $Z^{k-1} \mid \gcd(\mathcal{E}_1,\mathcal{E}_2)$.*

Let us consider the following GL-comitants of systems (2.1):

Notation 3.8.

$$
\begin{array}{ll}
M(a,x,y) = 2\,\mathrm{Hess}\,\big(C_2(x,y)\big), & \eta(a) = \mathrm{Discriminant}\,\big(C_2(x,y)\big),\\
K(a,x,y) = \mathrm{Jacob}\,\big(p_2(x,y), q_2(x,y)\big), & \mu(a) = \mathrm{Discriminant}\,\big(K(a,x,y)\big),\\
N(a,x,y) = K(a,x,y) + H(a,x,y), & \theta(a) = \mathrm{Discriminant}\,\big(N(a,x,y)\big),
\end{array}
$$

the geometrical meaning of which is revealed by the next 3 lemmas (see [24]).

Lemma 3.5. *Let $S \in \mathbf{QS}$ and let $\boldsymbol{a} \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. The common points of $P = 0$ and $Q = 0$ on the line $Z = 0$ are given by the common linear factors over $\mathbb{C}$ of p_2 and q_2. Moreover,*

$$
\deg \gcd(p_2(x,y), q_2(x,y)) = \begin{cases} 0 & iff \quad \mu(\boldsymbol{a}) \neq 0;\\ 1 & iff \quad \mu(\boldsymbol{a}) = 0,\ K(\boldsymbol{a},x,y) \neq 0;\\ 2 & iff \quad K(\boldsymbol{a},x,y) = 0. \end{cases}
$$

Lemma 3.6. *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a systems* (2.1) *corresponding to $\boldsymbol{a} \in \mathbb{R}^{12}$ is the condition $\theta(\boldsymbol{a}) = 0$ (respectively, $N(\boldsymbol{a},x,y) = 0$).*

Lemma 3.7. *The type of the divisor $D_S(C,Z)$ for systems (1.1) is determined by the corresponding conditions indicated in Table 1, where we write $\omega_1^c + \omega_2^c + \omega_3$ if two of the points, i.e. ω_1^c, ω_2^c, are complex but not real. Moreover, for each type of the divisor $D_S(C,Z)$ given by Table 1 the quadratic systems (1.1) can be brought via a linear transformation to one of the following canonical systems $(\mathbf{S}_I) - (\mathbf{S}_V)$ corresponding to their behavior at infinity.*

In order to determine the existence of a common factor of the polynomials $\mathcal{E}_1(\boldsymbol{a},X,Y,Z)$ and $\mathcal{E}_2(\boldsymbol{a},X,Y,Z)$ we shall use the notion of the resultant of two polynomials with respect to a given indeterminate (see for instance, [30]).

Let us consider two polynomials $f, g \in R[x_1, x_2, \ldots, x_r]$ where R is a unique factorization domain. Then we can regard the polynomials f and g as polynomials in x_r over the ring $R[x_1, x_2, \ldots, x_{r-1}]$, i.e.

$$
\begin{aligned}
f(x_1, x_2, \ldots, x_r) &= a_0 + a_1 x_r + \ldots + a_n x_r^n,\\
g(x_1, x_2, \ldots, x_r) &= b_0 + a_1 x_r + \ldots + b_m x_r^m.
\end{aligned}
$$

Lemma 3.8. [30] *Assuming $a_n b_m \neq 0$ and $n, m > 0$, the resultant $\mathrm{Res}_{\,x_r}(f,g)$ of the polynomials f and g with respect to x_r is a polynomial in $R[x_1, x_2, \ldots, x_{r-1}]$ which is zero if and only if f and g have a common factor involving x_r.*

Table 1

Case	Type of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$\omega_1+\omega_2+\omega_3$	$\eta>0$
2	$\omega_1^c+\omega_2^c+\omega_3$	$\eta<0$
3	$2\omega_1+\omega_2$	$\eta=0, \quad M\neq 0$
4	3ω	$M=0, \quad C_2\neq 0$
5	$D_S(C,Z)$ undefined	$C_2=0$

$$\left\{\begin{aligned} \frac{dx}{dt} &= k+cx+dy+gx^2+(h-1)xy,\\ \frac{dy}{dt} &= l+ex+fy+(g-1)xy+hy^2; \end{aligned}\right. \qquad (\mathbf{S}_I)$$

$$\left\{\begin{aligned} \frac{dx}{dt} &= k+cx+dy+gx^2+(h+1)xy,\\ \frac{dy}{dt} &= l+ex+fy-x^2+gxy+hy^2; \end{aligned}\right. \qquad (\mathbf{S}_{II})$$

$$\left\{\begin{aligned} \frac{dx}{dt} &= k+cx+dy+gx^2+hxy,\\ \frac{dy}{dt} &= l+ex+fy+(g-1)xy+hy^2; \end{aligned}\right. \qquad (\mathbf{S}_{III})$$

$$\left\{\begin{aligned} \frac{dx}{dt} &= k+cx+dy+gx^2+hxy,\\ \frac{dy}{dt} &= l+ex+fy-x^2+gxy+hy^2, \end{aligned}\right. \qquad (\mathbf{S}_{IV})$$

$$\left\{\begin{aligned} \frac{dx}{dt} &= k+cx+dy+x^2,\\ \frac{dy}{dt} &= l+ex+fy+xy. \end{aligned}\right. \qquad (\mathbf{S}_{V})$$

4 The Main Theorem

Notation 4.1. *We denote by* $\mathbf{QSL_4}$ *the class of all quadratic differential systems* (2.1) *with* $(p,q)=1$ *possessing a configuration of 4 invariant straight lines including the line at infinity and including possible multiplicities and the line at infinity does not consist entirely of singularities.*

Observation 4.2. *The case when the line at infinity is a union of singularities will be discussed in a forthcoming paper.*

Lemma 4.1. *If a quadratic system* (S) *corresponding to* $\boldsymbol{a}\in\mathbb{R}^{12}$ *belongs to the class* $\mathbf{QSL_4}$*, then for this system one of the following two conditions are satisfied in* $\mathbb{R}[x,y]$*:*

$(i)\quad \theta\neq 0,\ B_3(\boldsymbol{a},x,y)=0; \qquad (ii)\quad \theta(\boldsymbol{a})=0=B_2(\boldsymbol{a},x,y).$

Proof: Indeed, if for a system (2.1) the condition $M_{\mathbf{IL}}=4$ is satisfied then taking into account the Definition 1.2 we conclude that there exists a perturbation of the coefficients of the system (2.1)

within the class of quadratic systems such that the perturbed systems have 4 distinct invariant lines (real or complex, including the line $Z = 0$). Hence, the perturbed systems must possess either three affine lines with distinct directions or one couple of parallel lines and another line in a different direction. Then, by continuity and according to Lemmas 3.2 and 3.6 we respectively have either conditions (i) or (ii). ∎

By Lemmas 3.3, 3.4 and 3.6 we obtain the following result:

Lemma 4.2. *(a) If for a system (S) of coefficients $\boldsymbol{a} \in \mathbb{R}^{12}$, $M_{\mathbf{IL}} = 4$ then* $\deg\gcd\big(\mathcal{E}_1(\boldsymbol{a}, X, Y, Z), \mathcal{E}_2(\boldsymbol{a}, X, Y, Z)\big) = 3$; *(b) If $\theta(\boldsymbol{a}) \neq 0$ then $M_{\mathbf{IL}} \leq 4$.*

We shall use here the following T-comitants constructed in [24]

$$
\begin{aligned}
&H_1(a) = -\big((C_2, C_2)^{(2)}, C_2)^{(1)}, D\big)^{(3)},\\
&H_2(a, x, y) = (C_1,\ 2H - N)^{(1)} - 2D_1N,\\
&H_3(a, x, y) = (C_2, D)^{(2)},\\
&H_4(a) = \big((C_2, D)^{(1)}, (C_2, D_2)^{(1)}\big)^{(2)},\\
&H_5(a) = \big((C_2, C_2)^{(2)}, (D, D)^{(2)}\big)^{(2)} + 8\big((C_2, D)^{(2)}, (D, D_2)^{(1)}\big)^{(2)},\\
&H_6(a, x, y) = 16N^2(C_2, D)^{(2)} + H_2^2(C_2, C_2)^{(2)}
\end{aligned}
$$

and CT-comitants

$$
\begin{aligned}
&N_1(a, x, y) = C_1(C_2, C_2)^{(2)} - 2C_2(C_1, C_2)^{(2)},\\
&N_2(a, x, y) = D_1(C_1, C_2)^{(2)} - \Big((C_2, C_2)^{(2)}, C_0\Big)^{(1)},\\
&N_3 = (a, x, y)\,(C_2, C_1)^{(1)},\\
&N_4(a, x, y) = 4\,(C_2, C_0)^{(1)} - 3C_1D_1,\\
&N_5(a, x, y) = \big((D_2, C_1)^{(1)} + D_1D_2\big)^2 - 4\big(C_2, C_2\big)^{(2)}\big(C_0, D_2\big)^{(1)},\\
&N_6(a, x, y) = 8D + C_2\Big[8(C_0, D_2)^{(1)} - 3(C_1, C_1)^{(2)} + 2D_1^2\Big].
\end{aligned}
$$

We shall also use the following remark:

Remark 4.3. *Assume $s, \gamma \in \mathbb{R}$, $\gamma > 0$. Then the transformation $x = \gamma^s x_1$, $y = \gamma^s y_1$ and $t = \gamma^{-s} t_1$ does not change the coefficients of the quadratic part of a quadratic system, whereas each coefficient of the linear (respectively, constant) part will be multiplied by γ^{-s} (respectively, by γ^{-2s}).*

Theorem 4.1. *(i) The class* $\mathbf{QSL_4}$ *splits into 46 distinct subclasses indicated in* **Diagram 1** *with the corresponding Configurations 4.1-4.46 where the complex invariant straight lines are indicated by dashed lines. If an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. We indicate next to the singular points their multiplicities as follows: $(I_\omega(p, q))$ if ω is a finite singularity, $(I_\omega(C, Z),\ I_\omega(P, Q))$ if ω is an infinite singularity with $I_w(P, Q) \neq 0$ and $(I_\omega(C, Z))$ if ω is an infinite singularity with $I_\omega(P, Q) = 0$.*

(ii) We consider the orbits of the class $\mathbf{QSL_4}$ *under the action of the affine group and time rescaling. The systems of the form (IV.1) up to form (IV.46) from the Table 2 form a system of representatives of these orbits under this action. A differential system (S) in* $\mathbf{QSL_4}$ *is in the orbit of a system belonging to $(IV.i)$ if and only if the corresponding conditions in the middle column*

(where the polynomials H_i $(i = 7, \dots, 11)$ are T-comitants to be introduced below) is verified for this system (S). The conditions indicated in the middle column are affinely invariant.

Wherever we have a case with invariant straight lines of multiplicity > 1 we indicate the corresponding perturbed systems in the Table 3.

Proof of the Main Theorem: Since we only discuss the case $C_2 \neq 0$, in what follows it suffices to consider only the canonical forms $(\mathbf{S}_I)$ to $(\mathbf{S}_{IV})$. The idea of the proof is to perform a case by case discussion for each one of these canonical forms, for which according to Lemma 4.1 we must examine two subcases: (i) $\theta \neq 0$, $B_3 = 0$ and (ii) $\theta = B_2 = 0$. Each one of these conditions yields specific conditions on the parameters. The discussion proceeds further by breaking these cases in more subcases and then by constructing new invariants or T-comitants to fit the conditions on parameters.

4.1 Systems with the divisor $D_S(C, Z) = 1 \cdot \omega_1 + 1 \cdot \omega_2 + 1 \cdot \omega_3$

For this case we shall later need the following T-comitants.

Notation 4.4. *Let us denote*

$$H_7(a) = (N, C_1)^{(2)}, \quad H_9(\boldsymbol{a}) = -\Big(\big((D, D)^{(2)}, D,\big)^{(1)} D\Big)^{(3)},$$

$$H_8(\boldsymbol{a}) = 9\Big((C_2, D)^{(2)}, (D, D_2)^{(1)}\Big)^{(2)} + 2\Big[(C_2, D)^{(3)}\Big]^2,$$

$$H_{10}(\boldsymbol{a}) = \big((N, D)^{(2)},\ D_2\big)^{(1)}.$$

According to Lemma 3.7 the systems with this type of divisor can be brought by linear transformations to the canonical form $(\mathbf{S}_I)$ for which we have:

$$\theta(\boldsymbol{a}, x, y) = -8(g-1)(h-1)(g+h). \tag{4.1}$$

4.1.1 The case $\theta \neq 0$, $B_3 = 0$

The condition $\theta \neq 0$ yields $(g-1)(h-1) \neq 0$ and in $(\mathbf{S}_I)$ we may assume $d = e = 0$ via the translation $x \to x + d/(1-h)$ and $y \to y + e/(1-g)$. Thus we obtain the systems

$$\dot{x} = k + cx + gx^2 + (h-1)xy, \quad \dot{y} = l + fy + (g-1)xy + hy^2, \tag{4.2}$$

for which we calculate

$$\begin{aligned} B_3 = {} & 73l(g-1)^2x^3(2y-x) + 3k(h-1)^2y^3(y-2x) + \\ & 3[(c-f)(fg+ch) - k(1+g)(-1+g+2h) + l(1+h)(-1+2g+h)]x^2y^2. \end{aligned}$$

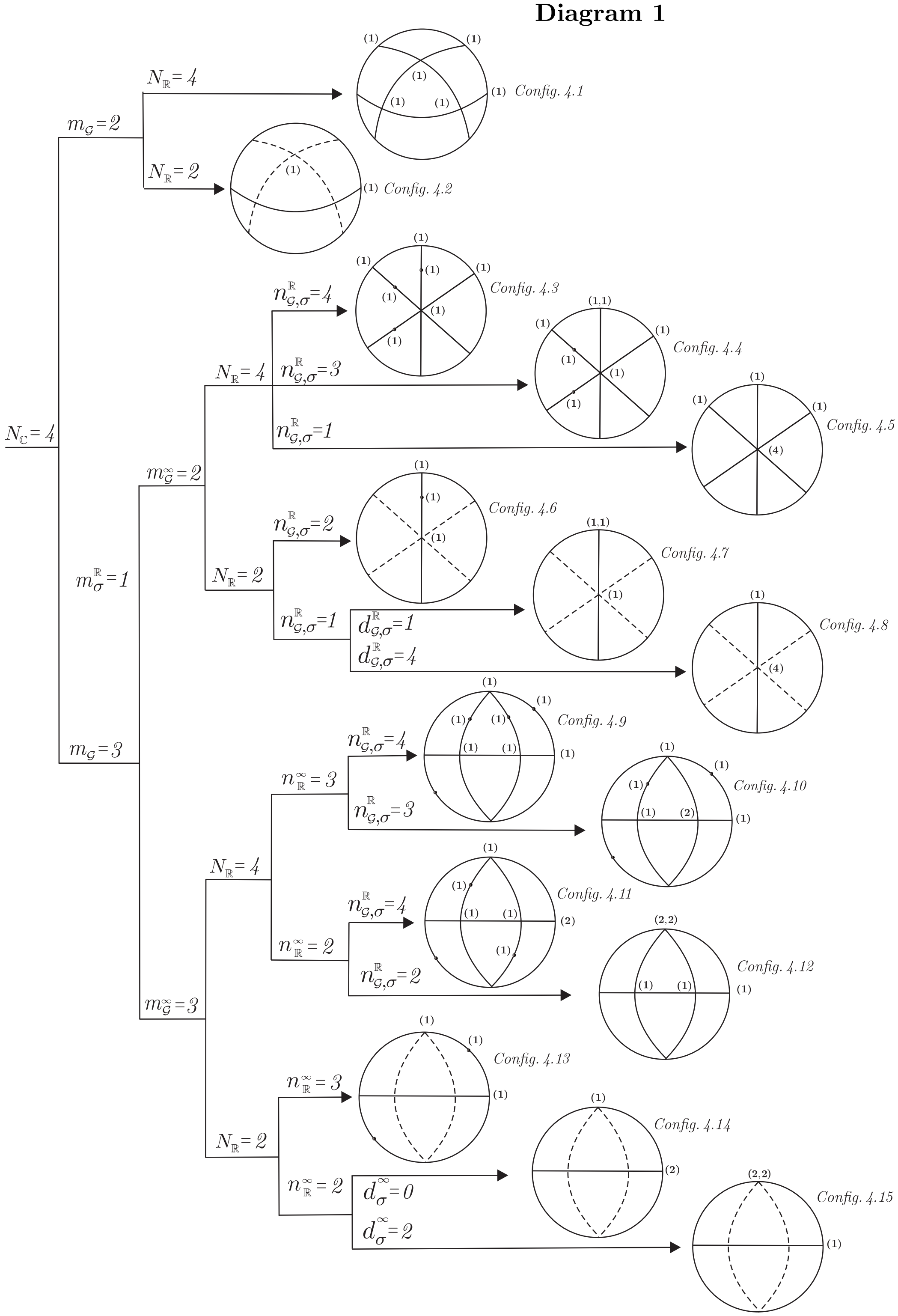
Then by $\theta \neq 0$ the condition $B_3 = 0$ yields $k = l = 0$ and $(c-f)(fg+ch) = 0$. Hence $c - f = 0$ or $fg + ch = 0$. An invariant which capture the condition $c - f = 0$ is H_7. Indeed, for the systems (4.2) we have $H_7 = 4(f-c)(g-1)(h-1)$ and since $\theta \neq 0$ the condition $f - c = 0$ is equivalent with $H_7 = 0$.

Subcase $H_7 = 0$. Then $f = c$ and we obtain the systems

$$\dot{x} = cx + gx^2 + (h-1)xy, \qquad \dot{y} = cy + (g-1)xy + hy^2, \tag{4.3}$$

for which calculations yield $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2XY(X-Y)$ in the ring $\mathbb{R}[c, g, h, X, Y, Z]$ which means that for a concrete system corresponding to $(\mathbf{c}, \mathbf{g}, \mathbf{h})$, $M_{\mathbf{IL}} \geq 4$ and by Lemma 4.2 $M_{\mathbf{IL}}$ cannot be 5.

Diagram 1



$N_{\mathbb{C}}=4$
$m_{\mathcal{G}}=2$
$N_{\mathbb{R}}=4$
$N_{\mathbb{R}}=2$
Config. 4.1
Config. 4.2
$m_{\sigma}^{\mathbb{R}}=1$
$m_{\mathcal{G}}^{\infty}=2$
$N_{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
Config. 4.3
Config. 4.4
Config. 4.5
$N_{\mathbb{R}}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.6
Config. 4.7
Config. 4.8
$m_{\mathcal{G}}=3$
$m_{\mathcal{G}}^{\infty}=3$
$N_{\mathbb{R}}=4$
$n_{\mathbb{R}}^{\infty}=3$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
$n_{\mathbb{R}}^{\infty}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
Config. 4.9
Config. 4.10
Config. 4.11
Config. 4.12
$N_{\mathbb{R}}=2$
$n_{\mathbb{R}}^{\infty}=3$
$n_{\mathbb{R}}^{\infty}=2$
$d_{\sigma}^{\infty}=0$
$d_{\sigma}^{\infty}=2$
Config. 4.13
Config. 4.14
Config. 4.15

Diagram 1 *(continued)*

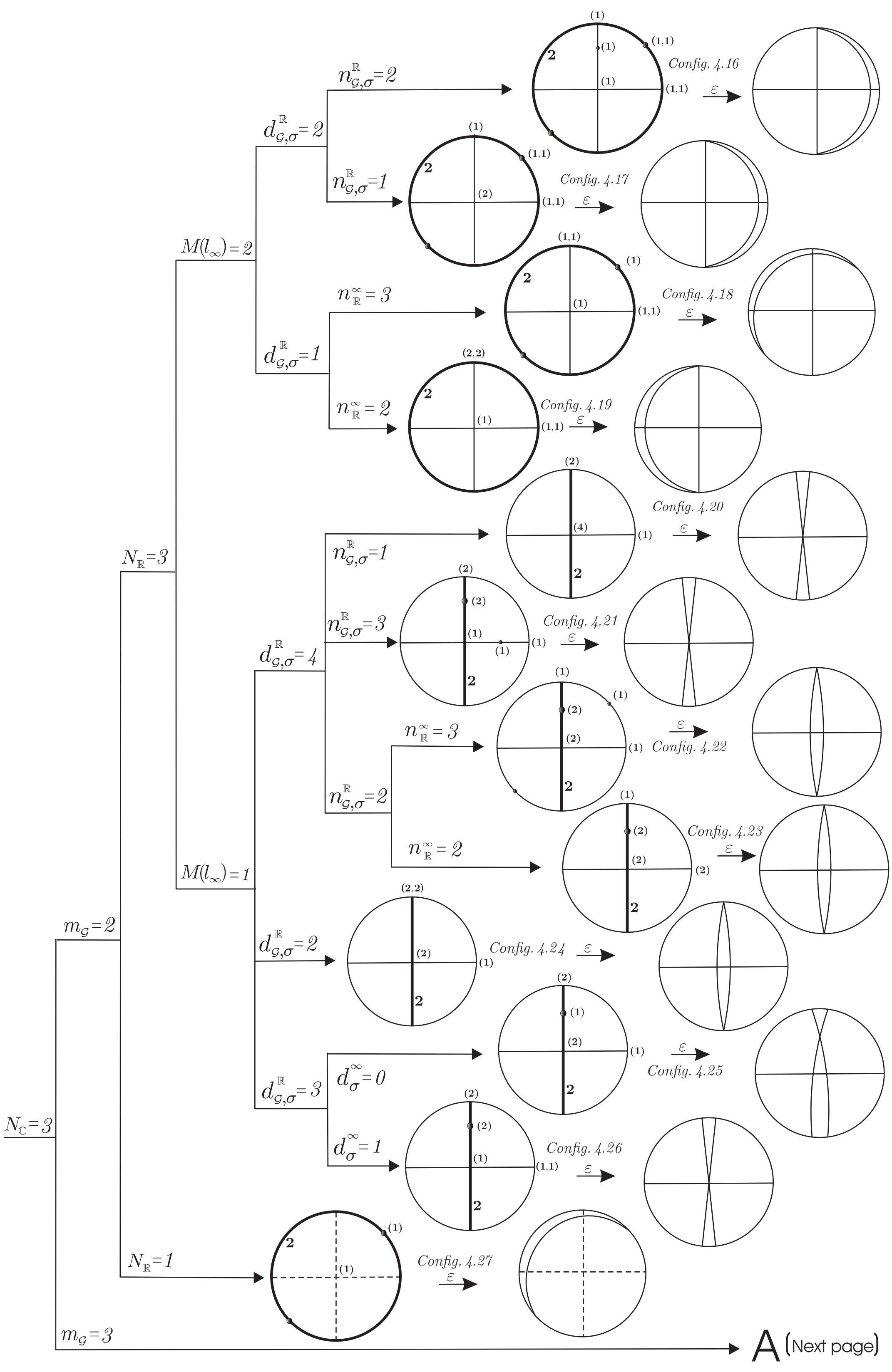


Diagram 1 *(continued)*

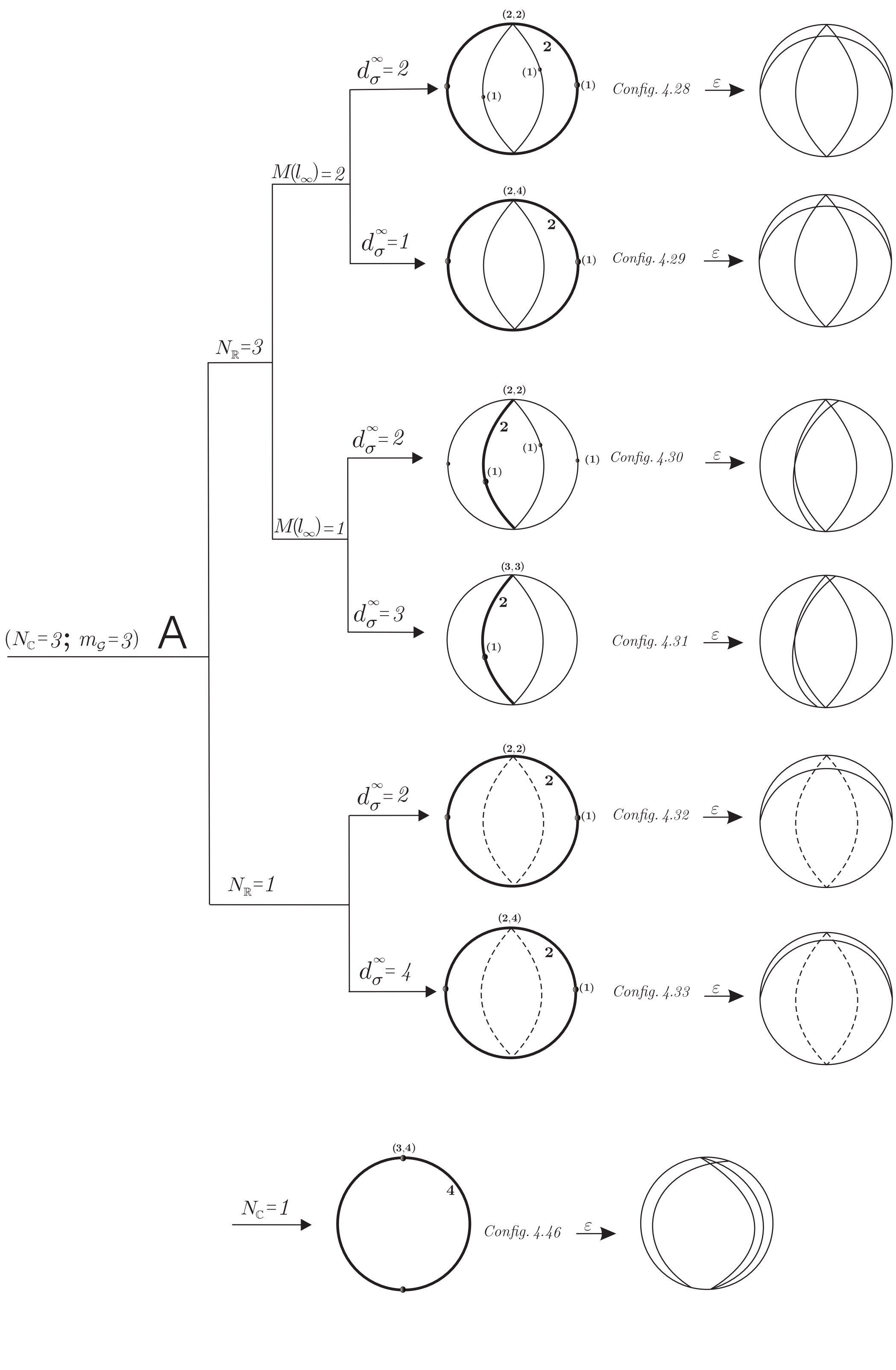


Diagram 1 *(continued)*

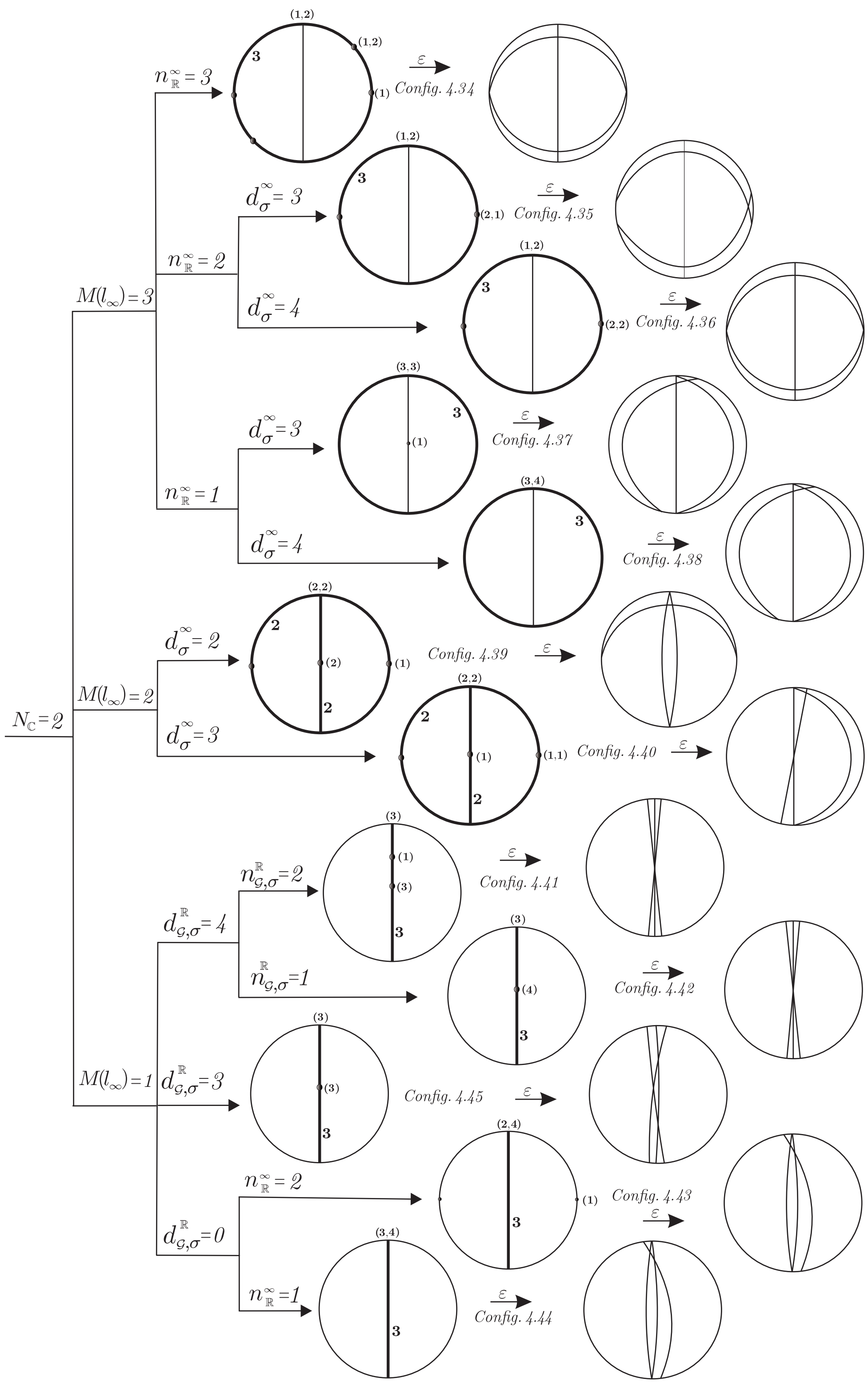


Table 2

<table>
<tr><th>Orbit representative</th><th colspan="2">Necessary and sufficient conditions</th><th>Configuration</th></tr>
<tr><td>$(IV.1)\begin{cases}\dot{x}=lx+lx^2+(b-1)xy,\\ \dot{y}=-by+(l-1)xy+by^2,\\ b,l\in\mathbb{R},\ bl(b+l-1)\neq0,\\ (b-1)(l-1)(b+l)\neq0\end{cases}$</td><td colspan="2">$\eta>0,\ \theta\neq0,\ B_3=0,$
$H_7\neq0$</td><td>Config. 4.1</td></tr>
<tr><td>$(IV.2)\begin{cases}\dot{x}=lx^2+(b+1)xy,\ b,l\in\mathbb{R}\\ \dot{y}=b[l^2+(b+1)^2]+2lby\\ +(l^2+1-h^2)x-x^2+lxy+by^2\\ b(b+1)\big[l^2+(b-1)^2\big]\neq0\end{cases}$</td><td colspan="2">$\eta<0,\ \theta\neq0,\ B_3=0,$
$H_7\neq0$</td><td>Config. 4.2</td></tr>
<tr><td>$(IV.3)\begin{cases}\dot{x}=x+lx^2+(b-1)xy,\\ \dot{y}=y+(l-1)xy+by^2,\\ b,l\in\mathbb{R},\ bl(b+l-1)\neq0,\\ (b-1)(l-1)(b+l)\neq0\end{cases}$</td><td colspan="2">$\eta>0,\ \theta\neq0,\ B_3=0,$
$H_7=0,\ H_1\neq0,\ \mu\neq0$</td><td>Config. 4.3</td></tr>
<tr><td>$(IV.4)\begin{cases}\dot{x}=x+lx^2-xy,\\ \dot{y}=y+(l-1)xy,\\ l\in\mathbb{R},\ l(l-1)\neq0\end{cases}$</td><td colspan="2">$\eta>0,\ \theta\neq0,\ B_3=0,$
$H_7=0,\ H_1\neq0,\ \mu=0$</td><td>Config. 4.4</td></tr>
<tr><td>$(IV.5)\begin{cases}\dot{x}=lx^2+(b-1)xy,\\ \dot{y}=(l-1)xy+by^2,\\ b,l\in\mathbb{R},\ bl(b+l-1)\neq0,\\ (b-1)(l-1)(b+l)\neq0\end{cases}$</td><td colspan="2">$\eta>0,\ \theta\neq0,\ B_3=0,$
$H_7=0,\ H_1=0$</td><td>Config. 4.5</td></tr>
<tr><td>$(IV.6)\begin{cases}\dot{x}=lx^2+(b+1)xy,\ b,l\in\mathbb{R}\\ \dot{y}=-1+lx+(b-1)y\\ -x^2+lxy+by^2\\ b(b+1)\big[l^2+(b-1)^2\big]\neq0\end{cases}$</td><td colspan="2">$\eta<0,\ \theta\neq0,\ B_3=0,$
$H_7=0,\ \mu\neq0,\ H_9\neq0$</td><td>Config. 4.6</td></tr>
<tr><td>$(IV.7)\begin{cases}\dot{x}=lx^2+xy,\quad l\in\mathbb{R}\\ \dot{y}=-1+lx-y-x^2+lxy\end{cases}$</td><td colspan="2">$\eta<0,\ \theta\neq0,\ B_3=0,$
$H_7=0,\ \mu=0$</td><td>Config. 4.7</td></tr>
<tr><td>$(IV.8)\begin{cases}\dot{x}=lx^2+(b+1)xy,\ b,l\in\mathbb{R}\\ \dot{y}=-x^2+lxy+by^2\\ b(b+1)\big[l^2+(b-1)^2\big]\neq0\end{cases}$</td><td colspan="2">$\eta<0,\ \theta\neq0,\ B_3=0,$
$H_7=0,\ \mu\neq0,\ H_9=0$</td><td>Config. 4.8</td></tr>
<tr><td rowspan="2">$(IV.9)\begin{cases}\dot{x}=l(x^2-1),\ l,b\in\mathbb{R},\\ \dot{y}=(y+b)[y+(l-1)x-b],\\ l(l-1)\neq0,\ (l-1)^2\neq4b^2,\\ (l+1)^2\neq4b^2\end{cases}$</td><td rowspan="2">$\eta>0,\ \theta=B_2=0$
$\mu\,B_3\,H_4\neq0,$
$H_7=0, H_9\neq0$</td><td>$H_{10}N>0,$</td><td rowspan="2">Config. 4.9</td></tr>
<tr><td>$N=0, H_8>0$</td></tr>
<tr><td rowspan="2">$(IV.10)\begin{cases}\dot{x}=(2l+1)(x^2-1),\\ \dot{y}=(y+l)(y+2lx-l),\\ l\in\mathbb{R},\ l(2l+1)\neq0\end{cases}$</td><td rowspan="2">$\eta>0,\ \theta=B_2=0$
$\mu\,B_3\,H_4\neq0,$
$H_7=H_9=0$</td><td>$H_{10}N>0,$</td><td rowspan="2">Config. 4.10</td></tr>
<tr><td>$N=0, H_8>0$</td></tr>
<tr><td>$(IV.11)\begin{cases}\dot{x}=x^2+xy,\ l\in\mathbb{R},\ l\neq\pm1\\ \dot{y}=(y+l)^2-1\end{cases}$</td><td colspan="2">$\eta=0,\ M\neq0,\ \theta=B_2=0,$
$B_3\mu\neq0,\ H_7=0,\ H_{10}>0$</td><td>Config. 4.11</td></tr>
<tr><td>$(IV.12)\begin{cases}\dot{x}=l\big[(x+b)^2-1\big],\\ \dot{y}=(l-1)xy,\ l,b\in\mathbb{R},\\ l(l-1)(b^2-1)\neq0\end{cases}$</td><td colspan="2">$\eta=0,\ M\neq0,\ \theta=B_3=0,$
$KH_6\neq0,\ \mu=H_7=0,\ H_{11}>0$</td><td>Config. 4.12</td></tr>
<tr><td rowspan="2">$(IV.13)\begin{cases}\dot{x}=l(x^2+1),\\ \dot{y}=(y+b)[y+(l-1)x-b],\\ l,b\in\mathbb{R},\ l(l-1)\neq0\end{cases}$</td><td rowspan="2">$\eta>0,\ \theta=B_2=0$
$\mu\,B_3\,H_4\neq0,$
$H_7=0$</td><td>$H_{10}N<0,$</td><td rowspan="2">Config. 4.13</td></tr>
<tr><td>$N=0, H_8<0$</td></tr>
<tr><td>$(IV.14)\begin{cases}\dot{x}=x^2+xy,\ l\in\mathbb{R}\\ \dot{y}=(y+l)^2+1\end{cases}$</td><td colspan="2">$\eta=0,\ M\neq0,\ \theta=B_2=0,$
$B_3\mu\neq0,\ H_7=0,\ H_{10}<0$</td><td>Config. 4.14</td></tr>
</table>

Table 2 *(continued)*

Orbit representative	Necessary and sufficient conditions	Configuration
$(IV.15)\ \begin{cases}\dot{x}=l\left[(x+b)^2+1\right],\\ \dot{y}=(l-1)xy,\ l,b\in\mathbb{R},\\ l(l-1)\neq 0\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=B_3=0,$ $KH_6\neq 0,\ \mu=H_7=0,\ H_{11}<0$	Config. 4.15
$(IV.16)\ \begin{cases}\dot{x}=l+x,\ \ l\in\mathbb{R},\\ \dot{y}=y(y-x),\ l(l-1)\neq 0\end{cases}$	$\eta>0,\ \theta=B_2=\mu=0,$ $B_3\neq 0,\ H_7=0,\ H_9\neq 0$	Config. 4.16
$(IV.17)\ \begin{cases}\dot{x}=x,\\ \dot{y}=y(y-x)\end{cases}$	$\eta>0,\ \theta=B_2=\mu=0,$ $B_3\neq 0,\ H_7=H_9=0,\ H_{10}\neq 0$	Config. 4.17
$(IV.18)\ \begin{cases}\dot{x}=l(l+1)+lx+y,\ l\in\mathbb{R},\\ \dot{y}=y(y-x),\ l(l+1)\neq 0\end{cases}$	$\eta>0,\ \theta=B_3=\mu=0,$ $H_7\neq 0$	Config. 4.18
$(IV.19)\ \begin{cases}\dot{x}=l+x,\ \ l\in\mathbb{R},\\ \dot{y}=-xy,\ \ l(l-1)\neq 0\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=B_3=K=0,$ $NH_6\neq 0,\ \mu=H_7=0,\ H_{11}\neq 0$	Config. 4.19
$(IV.20)\ \begin{cases}\dot{x}=lx^2+xy,\ l(l-1)\neq 0\\ \dot{y}=(l-1)xy+y^2,\ l\in\mathbb{R}\end{cases}$	$\eta=0,\ M\neq 0,\ \theta\neq 0,$ $B_3=H_7=D=0$	Config. 4.20
$(IV.21)\ \begin{cases}\dot{x}=lx^2+xy,\ l(l-1)\neq 0\\ \dot{y}=(y+1)(lx-x+y),\ l\in\mathbb{R}\end{cases}$	$\eta=0,\ M\neq 0,\ \theta\neq 0,$ $B_3=H_7=0,\ D\neq 0,\ \mu\neq 0$	Config. 4.21
$(IV.22)\ \begin{cases}\dot{x}=lx^2,\ l\in\mathbb{R},\ l(l-1)\neq 0\\ \dot{y}=(y+1)[y+(l-1)x-1],\end{cases}$	$\eta>0,\ \theta=B_2=0$ $\mu B_3 H_4\neq 0,$ $H_7=0$; $N\neq 0,\ H_{10}=0$ $N=0,\ H_8=0$	Config. 4.22
$(IV.23)\ \begin{cases}\dot{x}=x^2+xy,\\ \dot{y}=(y+1)^2\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=B_2=0,$ $B_3\mu\neq 0,\ H_7=0,\ H_{10}=0$	Config. 4.23
$(IV.24)\ \begin{cases}\dot{x}=l(x+1)^2,\ l\in\mathbb{R},\\ \dot{y}=(l-1)xy,\ l(l-1)\neq 0\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=B_3=0,$ $KH_6\neq 0,\ \mu=H_7=0,\ H_{11}=0$	Config. 4.24
$(IV.25)\ \begin{cases}\dot{x}=lx^2+xy,\ l\in\mathbb{R},\ l(l-1)\neq 0\\ \dot{y}=y+(l-1)xy+y^2\end{cases}$	$\eta=0,\ M\neq 0,\ \theta\neq 0,$ $B_3=0,\ H_7\neq 0$	Config. 4.25
$(IV.26)\ \begin{cases}\dot{x}=xy,\\ \dot{y}=(y+1)(y-x)\end{cases}$	$\eta=0,\ M\neq 0,\ \theta\neq 0,$ $B_3=H_7=0,\ D\neq 0,\ \mu=0$	Config. 4.26
$(IV.27)\ \begin{cases}\dot{x}=2lx+2y,\ \ l\in\mathbb{R}\\ \dot{y}=l^2+1-x^2-y^2\end{cases}$	$\eta<0,\ \theta=0,\ B_3=0,$ $N\neq 0,\ H_7\neq 0$	Config. 4.27
$(IV.28)\ \begin{cases}\dot{x}=x^2-1,\ \ l\in\mathbb{R},\\ \dot{y}=x+ly,\ l(l^2-4)\neq 0\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=\mu=N=B_3=0,$ $N_1N_2\neq 0,\ K=0,\ N_5>0,\ D\neq 0$	Config. 4.28
$(IV.29)\ \begin{cases}\dot{x}=x^2-1,\ \ l\in\mathbb{R},\\ \dot{y}=l+x,\ \ l\neq\pm 1\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=\mu=N=B_3=0,$ $N_1N_2\neq 0,\ K=0,\ N_5>0,\ D=0$	Config. 4.29
$(IV.30)\ \begin{cases}\dot{x}=(1+x)(1+lx),\\ \dot{y}=1+(l-1)xy,\\ l\in\mathbb{R},\ l(l^2-1)\neq 0\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=H_6=0,$ $NB_3\neq 0,\ \mu=0,\ K\neq 0,\ H_{11}\neq 0$	Config. 4.30
$(IV.31)\ \begin{cases}\dot{x}=x+x^2,\ \ l\in\mathbb{R},\\ \dot{y}=l-x^2+xy,\ l(l+1)\neq 0\end{cases}$	$\eta=M=0,\ \theta=B_3=0,$ $N_6N\neq 0,\ H_{11}\neq 0$	Config. 4.31
$(IV.32)\ \begin{cases}\dot{x}=x^2+1,\ \ l\in\mathbb{R},\\ \dot{y}=x+ly,\ \ l\neq 0\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=\mu=N=B_3=0,$ $N_1N_2\neq 0,\ K=0,\ N_5<0,\ D\neq 0$	Config. 4.32
$(IV.33)\ \begin{cases}\dot{x}=x^2+1,\\ \dot{y}=l+x,\ \ l\in\mathbb{R}\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=\mu=N=B_3=0,$ $N_1N_2\neq 0,\ K=0,\ N_5<0,\ D=0$	Config. 4.33
$(IV.34)\ \begin{cases}\dot{x}=l,\ l\in\{-1,1\}\\ \dot{y}=y(y-x)\end{cases}$	$\eta>0,\ \theta=B_2=\mu=0,$ $B_3\neq 0,\ H_7=H_9=H_{10}=0$	Config. 4.34

Table 2 *(continued)*

Orbit representative	Necessary and sufficient conditions	Configuration
$(IV.35)\begin{cases}\dot x = l+y,\ l\in\mathbb{R},\ l\neq 0\\ \dot y=-xy\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=B_3=0,$ $N\neq 0,\ \mu=0,\ H_7\neq 0$	Config. 4.35
$(IV.36)\begin{cases}\dot x = l,\quad l\in\{-1,1\},\\ \dot y=-xy\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=B_3=K=0,$ $NH_6\neq 0,\ \mu=H_7=0,\ H_{11}=0$	Config. 4.36
$(IV.37)\begin{cases}\dot x = l+x,\ b\in\mathbb{R},\ l\in\{0,1\},\\ \dot y=by-x^2,\quad b(b^2-1)\neq 0\end{cases}$	$\eta=M=0,\ \theta=B_3=N=0,$ $N_3D_1\neq 0,\ N_6\neq 0,\ D\neq 0$	Config. 4.37
$(IV.38)\begin{cases}\dot x = l+x,\ b\in\mathbb{R},\ l\in\{0,1\},\\ \dot y=b-x^2,\quad b-l^2\neq 0\end{cases}$	$\eta=M=0,\ \theta=B_3=N=0,$ $N_3D_1\neq 0,\ N_6\neq 0,\ D=0$	Config. 4.38
$(IV.39)\begin{cases}\dot x = x^2,\\ \dot y=x+y\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=\mu=N=B_3=0,$ $N_1N_2\neq 0,\ K=0,\ N_5=0$	Config. 4.39
$(IV.40)\begin{cases}\dot x = 1+x,\\ \dot y=1-xy\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=H_6=0,$ $NB_3\neq 0,\ \mu=0,\ K=0$	Config. 4.40
$(IV.41)\begin{cases}\dot x = lxy,\quad l\in\{-1,1\},\\ \dot y=y-x^2+lxy,\end{cases}$	$\eta=M=0,\ \theta\neq 0,$ $B_3=0,\ H_7=0,\ D\neq 0$	Config. 4.41
$(IV.42)\begin{cases}\dot x = lxy,\quad l\in\{-1,1\},\\ \dot y=-x^2+lxy,\end{cases}$	$\eta=M=0,\ \theta\neq 0,$ $B_3=0,\ H_7=D=0$	Config. 4.42
$(IV.43)\begin{cases}\dot x = lx^2,\ l\in\mathbb{R},\ l(l^2-1)\neq 0\\ \dot y=1+(l-1)xy,\end{cases}$	$\eta=0,\ M\neq 0,\ \theta=H_6=0,$ $NB_3\neq 0,\ \mu=0,\ K\neq 0,\ H_{11}=0$	Config. 4.43
$(IV.44)\begin{cases}\dot x = x^2,\quad l\in\{-1,1\},\\ \dot y=l-x^2+xy\end{cases}$	$\eta=M=0,\ \theta=B_3=0,$ $N_6N\neq 0,\ H_{11}=0$	Config. 4.44
$(IV.45)\begin{cases}\dot x = lxy,\quad l\in\{-1,1\},\\ \dot y=x-x^2+lxy,\end{cases}$	$\eta=M=0,\ \theta\neq 0,$ $B_3=0,\ H_7\neq 0$	Config. 4.45
$(IV.46)\begin{cases}\dot x = 1,\\ \dot y=y-x^2\end{cases}$	$\eta=M=0,\ \theta=B_3=N=0,$ $N_3D_1\neq 0,\ N_6=0$	Config. 4.46

Table 3

Perturbed systems	Invariant straight lines
$(IV.16_\varepsilon):\ \dot x=(l+x)(\varepsilon x+1),\quad \dot y=y(y-x)$	$y=0,\ x=-l,\ \varepsilon x=-1$
$(IV.17_\varepsilon):\ \dot x=x(\varepsilon x+1),\quad \dot y=y(y-x)$	$y=0,\ x=0,\ \varepsilon x=-1$
$(IV.18_\varepsilon):\begin{cases}\dot x=(l^2+l+lx+y)(\varepsilon x+1),\\ \dot y=y[(y-x)-\varepsilon y[l(l+1)(\varepsilon+1)+1-y]\end{cases}$	$y=0,\ \varepsilon x+1=0,$ $y-x(1-l\varepsilon)=(l+1)(l\varepsilon+1)$
$(IV.19_\varepsilon):\ \dot x=(l+x)(\varepsilon x+1),\quad \dot y=-xy$	$y=0,\ x=-l,\ \varepsilon x=-1$
$(IV.20_\varepsilon):\ \dot x=lx^2+(\varepsilon+1)xy,\ \dot y=(l-1)xy+y^2$	$x=0,\ y=0,\ x+\varepsilon y=0$
$(IV.21_\varepsilon):\ \dot x=\varepsilon x+lx^2+(\varepsilon+1)xy,\ \dot y=(y+1)(lx-x+y)$	$x=0,\ y=-1,\ x+\varepsilon y=-\varepsilon$
$(IV.22_\varepsilon):\ \dot x=l(x^2-\varepsilon^2),\ \dot y=(y+1)[y+(l-1)x-1]$	$y+1=0,\ x=\pm\varepsilon$
$(IV.23_\varepsilon):\ \dot x=x^2+xy,\quad \dot y=(y+l)^2-\varepsilon^2$	$x=0,\quad y+l=\pm\varepsilon$
$(IV.24_\varepsilon):\ \dot x=l(x+1)^2-l\varepsilon^2,\quad \dot y=(l-1)xy$	$y=0,\quad x+1=\pm\varepsilon$
$(IV.25_\varepsilon):\begin{cases}\dot x=\varepsilon lx+lx^2+(\varepsilon+1)xy,\\ \dot y=y+(l-1)xy+y^2\end{cases}$	$x=0,\ y=0,\ x+\varepsilon y=-\varepsilon$
$(IV.26_\varepsilon):\ \dot x=\varepsilon x+(\varepsilon+1)xy,\quad \dot y=(y+1)(y-x)$	$x=0,\ y=-1,\ x+\varepsilon y=-\varepsilon$

Table 3 *(continued)*

Perturbed systems	Invariant straight lines
$(IV.27_\varepsilon): \begin{cases} \dot x = 2(1-2\varepsilon)(lx+y)(1+\varepsilon x), \\ \dot y = l^2+1+2(l^2+1)\varepsilon x \\ \quad +(1-2\varepsilon)\left[-x^2+2l\varepsilon xy-(1-2\varepsilon)y^2\right] \end{cases}$	$\varepsilon x+1=0,$ $(1-2\varepsilon)(x\pm iy)=1\mp il$
$(IV.28_\varepsilon): \ \dot x = x^2-1, \quad \dot y=(x+ly)(1+\varepsilon y)$	$x=\pm1, \quad \varepsilon y=-1$
$(IV.29_\varepsilon): \ \dot x = x^2-1, \quad \dot y=(l+x)(1+\varepsilon y)$	$x=\pm1, \quad \varepsilon y=-1$
$(IV.30_\varepsilon): \begin{cases} \dot x=(1+x)(1+lx)-\varepsilon, \\ \dot y=1+(l-1)xy-\varepsilon y^2 \end{cases}$	$x+\varepsilon y+1=0$ $lx^2+(l+1)x+1=\varepsilon$
$(IV.31_\varepsilon): \begin{cases} \dot x=-l\varepsilon+(1+\varepsilon)x+(1+\varepsilon)x^2, \\ \dot y=l-x^2+xy \end{cases}$	$x+\varepsilon y=-1-\varepsilon,$ $(1+\varepsilon)x^2+(1+\varepsilon)x=l\varepsilon$
$(IV.32_\varepsilon): \ \dot x = x^2+1, \quad \dot y=(x+ly)(1+\varepsilon y)$	$x=\pm i, \quad \varepsilon y=-1$
$(IV.33_\varepsilon): \ \dot x = x^2+1, \quad \dot y=(l+x)(1+\varepsilon y)$	$x=\pm i, \quad \varepsilon y=-1$
$(IV.34_\varepsilon): \ \dot x = l(1-\varepsilon^2x^2), \quad \dot y=y(y-x)$	$y=0, \quad \varepsilon x=\pm1$
$(IV.35_\varepsilon): \begin{cases} \dot x=(\varepsilon x+1)[(\varepsilon+1)(y+l)+l\varepsilon x], \\ \dot y=l\varepsilon^2y+(l\varepsilon^3-1)xy+\varepsilon^2y^2 \end{cases}$	$y=0,\ \varepsilon x=-1,$ $\varepsilon^2(x+\varepsilon y)+\varepsilon=-1$
$(IV.36_\varepsilon): \ \dot x = l(1-\varepsilon^2x^2), \quad \dot y=-xy$	$y=0, \quad \varepsilon x=\pm1$
$(IV.37_\varepsilon): \begin{cases} \dot x=l+x+\varepsilon(2-l\varepsilon)x^2, \\ \dot y=by-x^2+\varepsilon(1+b-2l\varepsilon)xy+\varepsilon^2(b-l\varepsilon)y^2 \end{cases}$	$\varepsilon x+\varepsilon^2y+1=0,$ $\varepsilon(2-l\varepsilon)x^2+x+l=0$
$(IV.38_\varepsilon): \begin{cases} \dot x=l+x+\varepsilon(2-l\varepsilon+b\varepsilon^2)x^2, \\ \dot y=b-x^2+\varepsilon(1-2l\varepsilon-2b\varepsilon^2)xy-\varepsilon^3(l+b\varepsilon)y^2 \end{cases}$	$\varepsilon x+\varepsilon^2y+1=0,$ $\varepsilon(2-l\varepsilon+b\varepsilon^2)x^2+x=-l$
$(IV.39_\varepsilon): \ \dot x = x^2-\varepsilon^2, \quad \dot y=(x+ly)(1+\varepsilon y)$	$x=\pm\varepsilon, \quad \varepsilon y=-1$
$(IV.40_\varepsilon): \begin{cases} \dot x=(1+x)(1+\varepsilon x)-\varepsilon, \\ \dot y=1+(\varepsilon-1)xy-\varepsilon y^2 \end{cases}$	$x+\varepsilon y+1=0$ $\varepsilon x^2+(\varepsilon+1)x+1=\varepsilon$
$(IV.41_\varepsilon): \begin{cases} \dot x=l\varepsilon x^2/2+lxy, \\ \dot y=\varepsilon^2+2\varepsilon x+(1+2l\varepsilon^2)y-x^2+2l\varepsilon xy+l(1+l\varepsilon^2)y^2 \end{cases}$	$x=0,\ 2x+l\varepsilon y=-\varepsilon,$ $x-2l\varepsilon y=2\varepsilon$
$(IV.42_\varepsilon): \begin{cases} \dot x=\varepsilon x^2/2+lxy, \\ \dot y=-x^2+2\varepsilon xy+(l+\varepsilon^2)y^2 \end{cases}$	$x=0,\ 2x+\varepsilon y=0,$ $x-2\varepsilon y=0$
$(IV.43_\varepsilon): \ \dot x = l(x^2-\varepsilon^2), \quad \dot y=1+(l-1)xy-l\varepsilon^2y^2$	$x=\pm\varepsilon,\ x+l\varepsilon^2y=0$
$(IV.44_\varepsilon): \begin{cases} \dot x=-l\varepsilon+\varepsilon(1+\varepsilon)x+(1+\varepsilon)x^2, \\ \dot y=l-x^2+xy \end{cases}$	$x+\varepsilon y=-\varepsilon(1+\varepsilon),$ $(1+\varepsilon)x^2+\varepsilon(1+\varepsilon)x=l\varepsilon$
$(IV.45_\varepsilon): \begin{cases} \dot x=\varepsilon(\varepsilon^2-l)x/l+\varepsilon x^2+lxy, \\ \dot y=x+\varepsilon(2\varepsilon^2-l)y/l-x^2-2\varepsilon xy+(l-2\varepsilon^2)y^2 \end{cases}$	$x=0,\ x+\varepsilon y=0,$ $x+2\varepsilon y=2\varepsilon^2/l$
$(IV.46_\varepsilon): \begin{cases} \dot x=1+\varepsilon x+\varepsilon x^2, \\ \dot y=y-x^2+\varepsilon(1-\varepsilon)xy+\varepsilon^2(l-\varepsilon)y^2 \end{cases}$	$\varepsilon x+\varepsilon^2y+1=0,$ $\varepsilon x^2+\varepsilon x+1=0$

Let us examine the singularities of systems (4.3). Clearly for $c=0$ the point $(0,0)$ will be of the multiplicity four and since for these systems we have $H_1(\boldsymbol{a})=576c^2$ the condition $c=0$ is equivalent to $H_1=0$. Hence, for $H_1=0$ we obtain Config. 4.5.

Consider now $H_1\neq0$ (i.e. $c\neq0$). By Remark 4.3 ($\gamma=c,\ s=1$) we may assume $c=1$ and then for $gh(g+h-1)\neq0$ the systems (4.3) possess the following finite singular points:

$$(0,\,0);\quad (0,\,-1/h);\quad (-1/g,\,0);\quad \big(1/(1-g-h),\ 1/(1-g-h)\big).$$

On the other hand for systems (4.3) we have $\mu=32gh(g+h-1)$ and hence for $\mu\neq0$ we get the Config. 4.3.

Consider now $\mu=0$ for which we obtain $gh(g+h-1)=0$ and without loss of generality we may assume $h=0$. Indeed, if $g=0$ (respectively, $g+h-1=0$) we can apply the linear transformation which will replace the straight line $x=0$ with $y=0$ (respectively, $x=0$ with

$y = x$) which reduces the case $g = 0$ (respectively, $g + h - 1 = 0$) to the case $h = 0$. In this case from (4.1) we have $\theta = 8g(g-1) \neq 0$ and hence we obtain Config. 4.4.

Subcase $H_7 \neq 0$**.** Then $c - f \neq 0$ and hence, the condition $B_3 = 0$ yields $fg + ch = 0$. If $g = 0$ then $ch = 0$. In this case $\theta = 8h^2 \neq 0$ hence we must have $c = 0$ yielding degenerate systems. So, $g \neq 0$ and by introducing a new parameter u, $c = gu$ we obtain $f = -hu$. Then the condition $H_7 = -4(g-1)(h-1)(g+h)u \neq 0$ implies $u \neq 0$ and we may assume $u = 1$ via Remark 4.3 ($\gamma = u$, $s = 1$). This leads to the systems:

$$\dot{x} = gx + gx^2 + (h-1)xy, \qquad \dot{y} = -hy + (g-1)xy + hy^2, \tag{4.4}$$

for which calculations yield $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2XY(X - Y + Z)$ in the ring $\mathbb{R}[g, h, X, Y, Z]$ which means that for a concrete system corresponding to $\mathbf{g}$, $\mathbf{h}$, $M_{\mathbf{IL}} \geq 4$ and by Lemma 4.2 $M_{\mathbf{IL}}$ cannot be 5. The singularities of systems (4.4) are

$$(0,0); \quad (0,1); \quad (-1,0); \quad (-h,g).$$

The point $(-h, g) \notin \mathcal{G}$ (otherwise the systems (4.4) become degenerate). So we obtain the Config 4.4.

4.1.2 The case $\theta = B_2 = 0$

According to (4.1) the condition $\theta = 0$ yields $(g-1)(h-1)(g+h) = 0$ and without loss of generality we can consider $h = 1$. Indeed, if $g = 1$ (respectively, $g + h = 0$) we can apply the linear transformation which will replace the straight line $x = 0$ with $y = 0$ (respectively, $x = 0$ with $y = x$) reducing this case to $h = 1$. Assuming $h = 1$ for the systems $(\mathbf{S}_I)$ we calculate $N = (g^2 - 1)x^2$ (see Lemma 3.6) and we shall examine two subcases: $N \neq 0$ and $N = 0$.

$\boxed{\textbf{Subcase } N \neq 0.}$ Then $(g-1)(g+1) \neq 0$. For systems $(\mathbf{S}_I)$ with $h = 1$ we have $\mu = 32g^2$ and we shall consider two subcases: $\mu \neq 0$ and $\mu = 0$.

1) If $\mu \neq 0$ then $g \neq 0$ and we may assume $c = f = 0$ via the translation $x \to x - c/(2g)$, $y \to y + \big[c(g-1) - 2fg\big]/(4g)$. Thus we obtain the systems

$$\dot{x} = k + dy + gx^2, \quad \dot{y} = l + ex + (g-1)xy + y^2, \tag{4.5}$$

for which $H_7 = 4d(g^2 - 1)$. We claim that a given system $S(\boldsymbol{a})$ of the form (4.5) to belong to the class $\mathbf{QSL_4}$ $\mathbf{d} = 0$ (i.e. $H_7(\boldsymbol{a}) = 0$) is necessary. Indeed, for systems (4.5) we calculate

$$C_2 = xy(x-y), \quad H = -x\big[(g-1)^2x + 4gy\big]$$

and hence there exist 3 directions (u, v) for invariant lines $ux + vy + w = 0$, and namely: $(1,0)$, $(0,1)$ and $(1,-1)$. Moreover, according to Lemma 3.1 the parallel lines could be only in the directions given by the T-comitant H, i.e. $(1,0)$ or $((g-1)^2, 4g)$. However since $N \neq 0$ (i.e. $g^2 - 1 \neq 0$) we obtain $g - 1 \neq 0$ and $(g-1)^2 \neq -4g$ (otherwise $0 = (g-1)^2 + 4g = (g+1)^2 \neq 0$). Thus the parallel lines could only be in the direction $(1,0)$ and we conclude that to have $M_{\mathbf{IL}} = 4$ for a system $S(\boldsymbol{a})$, the existence of at least one line in the direction $(1,0)$ is necessary, i.e. of a line with equation $x + \alpha = 0$. Calculations yield $d = 0$ in this case and our claim is proved.

Thus we assume $H_7 = 0$ (i.e. $d = 0$). Then for systems (4.5) we calculate:

$$B_2 = -648\big[e^2 + l(g-1)^2\big]\big[e^2 + (l-k)(g+1)^2\big]$$

and the condition $B_2 = 0$ yields either

$$(i) \quad e^2 + l(g-1)^2 = 0 \qquad \text{or} \qquad (ii) \quad e^2 + (l-k)(g+1)^2 = 0.$$

We claim that the case (i) can be reduced by a linear transformation and time rescaling to the case (ii) and viceversa. Indeed, via the transformation $x_1 = x$, $y_1 = x - y$ and $t_1 = -t$ systems (4.5) with $d = 0$ keep the same form

$$\dot{x}_1 = \tilde{k} + \tilde{g}x_1^2, \quad \dot{y} = \tilde{l} + \tilde{e}x_1 + (\tilde{g} - 1)x_1y_1 + y_1^2$$

but with new parameters: $\tilde{k} = -k$, $\tilde{g} = -g$, $\tilde{l} = l - k$, and $\tilde{e} = e$. Then obviously we have:

$$\tilde{e}^2 + \tilde{l}(\tilde{g} - 1)^2 = e^2 + (l - k)(g + 1)^2, \qquad \tilde{e}^2 + (\tilde{l} - \tilde{k})(\tilde{g} + 1)^2 = e^2 + l(g - 1)^2$$

and this proves our claim.

In what follows we assume that the condition (i) holds. Since $g - 1 \neq 0$ we may set $e = u(g-1)$ (where u is a new parameter) and then we obtain $l = -u^2$. So we get the systems

$$\dot{x} = k + gx^2, \quad \dot{y} = (y + u)\big[y + (g - 1)x - u\big] \tag{4.6}$$

which posses three invariant straight lines: $y + u = 0$ and $gx^2 + k = 0$. So for a concrete system in this family $M_{\mathbf{IL}} \geq 4$. We shall find the conditions on the parameters, for systems (4.6) to be in the class $\mathbf{QSL_4}$, i.e. to possess exactly 4 invariant lines, including the line at infinity and including multiplicities. Calculations yield

$$\begin{aligned}
\mathcal{E}_1 &= 2\big[(g - 1)X^2 - (g - 3)XY - 2Y^2 - u(g + 1)XZ + 2uYZ - k(g - 1)Z^2\big]\mathcal{H},\\
\mathcal{E}_2 &= \big[(g - 1)X + Y - uZ\big]\big[gX^2 + (1 - g)XY - Y^2 + u(1 - g)XZ + (k + u^2)Z^2\big]\mathcal{H},
\end{aligned}$$

where $\mathcal{H} = (Y + uZ)(gX^2 + kZ^2) \in \mathbb{R}[g, k, u, X, Y, Z]$. The condition on the parameters g, k, u so as to have an additional common factor of $\mathcal{E}_1$, $\mathcal{E}_2$ according to Lemma 3.8 is

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{F},\ \mathcal{E}_2/\mathcal{F}) = 8(1 - g)\big[4gu^2 + k(g + 1)^2\big]\big[gY^2 - 2guYZ + (k(g - 1)^2 + gu^2)Z^2\big]^2 Z^2 \equiv 0$$

in $\mathbb{R}[X, Y, Z]$. Thus, since $g(g - 1) \neq 0$ the condition $\operatorname{Res}_X \equiv 0$ is equivalent to the condition $4gu^2 + k(g + 1)^2 = 0$. So the condition for $(S) \in \mathbf{QSL_4}$ is $4gu^2 + k(g + 1)^2 \neq 0$ which in view of

$$B_3 = -3[4gu^2 + k(g + 1)^2]x^2y^2$$

means $B_3 \neq 0$.

It remain to examine in more details the invariant lines configuration of systems (4.6). We observe that the lines depends on gk. For these systems we have

$$H_{10} = -32gk(g^2 - 1), \qquad N = (g^2 - 1)x^2.$$

So $H_{10}N = -32gk(g^2 - 1)^2x^2$. Since $g^2 - 1 \neq 0$ for a real value $x \neq 0$, $H_{10}N = -gk\Lambda$ with $\Lambda > 0$.

a) Assume $H_{10}N > 0$. Then $gk < 0$ and since $\mu \neq 0$ (i.e. $g \neq 0$) we may set $k = -gv^2$, where $v \neq 0$ is a new parameter. We may assume $v = 1$ via Remark 4.3 ($\gamma = v$,$s = 1$) and systems (4.6) become

$$\dot{x} = g(x^2 - 1), \quad \dot{y} = (y + u)\big[y + (g - 1)x - u\big] \tag{4.7}$$

with the singularities:

$$(1, -u), \quad (-1, -u), \quad (1, u - g + 1), \quad (-1, u + g - 1).$$

So among these there can be double points if and only if either $g - 1 = 2u$ or $g - 1 = -2u$. On the other hand for systems (4.7) we have $H_9 = 2^{10}3^2g^6\big[(g - 1)^2 - 4u^2\big]^2$ and hence, for $H_9 \neq 0$ we get Config. 4.9 whereas for $H_9 = 0$ we obtain Config 4.6. In the last case we may choose the

relation $g-1=2u$ as $g-1=-2u$ leads to this relation via the change $x\to -x$, $y\to -y$ and $t\to -t$.

b) Suppose now $H_{10}N<0$, i.e. $gk>0$. The systems (4.6) possess 2 imaginary affine parallel invariant lines. As above we may set $k=gv^2$ and assuming $v=1$ by Remark 4.3 ($\gamma=v$,$s=1$), we obtain the systems

$$\dot{x}=g(x^2+1),\quad \dot{y}=(y+u)\big[y+(g-1)x-u\big] \tag{4.8}$$

without real finite singularities. Thus we get the Config. 4.13.

c) For $H_{10}=0$, since $g\neq 0$ we obtain $k=0$ and for systems (4.6) the invariant line $x=0$ is a double line. Since in this case $B_3=-12gu^2x^2y^2\neq 0$ we can assume $u=1$ via the Remark 4.3 ($\gamma=u$,$s=1$). Thus we obtain the systems

$$\dot{x}=gx^2,\quad \dot{y}=(y+1)\big[y+(g-1)x-1\big]$$

and this leads to the Config. 4.22.

2) Assume $\mu=0$. Then $g=0$ and we may consider $e=f=0$ via the translation $x\to x+2e+f$, $y\to y+e$. So the systems $(\mathbf{S}_I)$ with $\theta=\mu=0$ (i.e. $h=1$, $g=0$) become

$$\dot{x}=k+cx+dy,\qquad \dot{y}=l-xy+y^2 \tag{4.9}$$

for which $B_2=-648l(c^2+cd-k+l)x^4=0$. Hence the condition $B_2=0$ yields either (i) $l=0$ or (ii) $c^2+cd-k+l=0$.

We claim that the case (ii) can be reduced by an affine transformation and time rescaling to the case (i) and viceversa. Indeed, via the transformation

$$x_1=x+2c+d,\qquad y_1=x-y+c+d\qquad t_1=-t$$

the systems (4.9) keep the same form

$$\dot{x}_1=\tilde{k}+\tilde{c}x_1+\tilde{d}y_1,\qquad \dot{y}_1=\tilde{l}-x_1y_1+y_1^2$$

but with new parameters:

$$\tilde{k}=2c(c+d)-k,\quad \tilde{c}=-(c+d),\quad \tilde{l}=l-k+c(c+d),\quad \tilde{d}=d.$$

Then obviously we have:

$$\tilde{l}=c^2+cd-k+l,\qquad \tilde{c}^2+\tilde{c}\tilde{d}-\tilde{k}+\tilde{l}=l$$

and this proves our claim.

Hence we only need to consider $l=0$. For systems (4.9) we have $H_7=-4d$ and we shall consider two subcases: $H_7\neq 0$ and $H_7=0$.

a) Consider first the case $H_7\neq 0$. Then $d\neq 0$ and we may assume $d=1$ via Remark 4.3 ($\gamma=d$, $s=1$). For systems (4.9) with $l=0$ and $d=1$ calculations yield:

$$\begin{aligned}\mathcal{E}_1&=\big[-2Y^3+Y^2F_1(X,Z)+YF_2(X,Z)+F_3(X,Z)\big]\mathcal{H},\\ \mathcal{E}_2&=(Y-X)(cX+Y+kZ)(XY-Y^2+cXZ+YZ+kZ^2)\mathcal{H},\end{aligned}$$

where $\mathcal{H}=YZ\in\mathbb{R}[c,k,X,Y,Z]$ and $F_i\in\mathbb{R}[c,k,X,Z]$ $(i=1,2,3)$ are homogeneous in X,Z of degree i. Hence deg $\mathcal{H}=2$ and for a system $S(\boldsymbol{a})$ to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ according to Lemma 3.8 it is necessary that the following identity holds in $\mathbb{R}[X,Z]$:

$$\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_1/\mathcal{H})=-16(c^2+c-k)(cX+kZ)^2(cX+X+kZ)^6Z^4=0.$$

So, we obtain the condition $(c^2+c-k)(c^2+k^2)((c+1)^2+k^2)=0$. However, for $k=c(c+1)=0$ we get a degenerate system. On the other hand for systems (4.9) with $l=0$ and $d=1$ we calculate $B_3=3(c^2+c-k)x^2y^2$ and hence, to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ it is necessary that $B_3=0$. Then $k=c(c+1)$ and we get the systems

$$\dot{x}=c(c+1)+cx+y, \qquad \dot{y}=-xy+y^2 \tag{4.10}$$

for which we obtain $\mathcal{H}=\gcd(\mathcal{E}_1,\mathcal{E}_2)=YZ(X-Y+cZ+Z)$, i.e. $\deg\mathcal{H}=3$.

We claim that these systems belong to class $\mathbf{QSL_4}$ or they are degenerate. Indeed, for systems (4.10) calculations yield:

$$\begin{aligned}
\mathcal{E}_1&=\big(-cX^2+2cXY+Y^2+2c^2YZ+2cYZ+c^3Z^2+c^2Z^2\big)\mathcal{H},\\
\mathcal{E}_2&=-3(X-Y)(Y+cZ)(cX+Y+c^2Z+cZ)\mathcal{H}.
\end{aligned}$$

Hence by Lemma 3.8 to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ the condition

$$\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=9c^2(c+1)^2(X+cZ)^6=0$$

must hold in $\mathbb{R}[X,Z]$. Therefore $c=0$ or $c=-1$. Both cases lead to degenerate systems (4.10). Our claim is proved.

We observe that systems (4.10) possess the two invariant affine straight lines: $y=0$ and $y=x+c+1$. Taking into account that $Z\mid\mathcal{H}$, we have by Corollary 3.7 that the line $Z=0$ could be of multiplicity 2. This is confirmed by the perturbations ($IV.18_\varepsilon$) from Table 3. On the other hand the systems (4.10) have two finite singularities: $(-c-1,0)$ and $(-c,-c)$ with $c\neq 0$ (otherwise we get a degenerate system). Thus, we get Config. 4.18.

b) Assume now $H_7=0$. Then $d=0$ and we get the systems

$$\dot{x}=k+cx, \qquad \dot{y}=-xy+y^2 \tag{4.11}$$

for which we obtain

$$\mathcal{E}_1=2(-X^2+3XY-2Y^2+cYZ+kZ^2)\mathcal{H}, \quad \mathcal{E}_2=(Y-X)(XY-Y^2+cXZ+kZ^2)\mathcal{H}.$$

where $\mathcal{H}=YZ(cX+kZ)=\gcd(\mathcal{E}_1,\mathcal{E}_2)$ and due to Lemma 3.8 in order to have for a specific system an additional nontrivial common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ it is necessary that the following condition holds in $\mathbb{R}[Y,Z]$:

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=4(c^2-k)(cY+kZ)^2Z^4=0,$$

i.e. $c^2-k=0$ or $c=k=0$. The second condition leads to degenerate system. Since for systems (4.11) we have $B_3=3(c^2-k)x^2y^2$ evidently the condition $\operatorname{Res}_X\not\equiv 0$ is equivalent to $B_3\neq 0$. We observe, that these systems possess 2 invariant affine lines $y=0$ and $cx+k=0$ for $c\neq 0$. Taking into account the value of $\mathcal{H}$, we have by Corollary 3.7 that the line $Z=0$ could be of multiplicity two and even three if $c=0$. This is confirmed by the perturbations ($IV.16_\varepsilon$) and ($IV.34_\varepsilon$) from Table 3. It remains to note that in the case $c\neq 0$ the systems (4.11) possess the finite singular points $(-k/c,0)$ and $(-k/c,-k/c)$. Moreover, these points are distinct for $k\neq 0$ and they coincide for $k=0$. These points lie on the invariant lines $y=0$ and $cx+k=0$.

Thus, since for the systems (4.11) $H_{10}=-8c^2$ and $H_9=-576c^4k^2$, we obtain: Config. 4.16 for $H_9\neq 0$, Config. 4.17 for $H_9=0$, $H_{10}\neq 0$ and Config. 4.34 for $H_9=H_{10}=0$. We note that for $c\neq 0$ (respectively, $c=0$, $k\neq 0$) we may assume $c=1$ (respectively, $k\in\{-1,1\}$) via Remark 4.3 ($\gamma=c$,$s=1$) (respectively, $\gamma=|k|$,$s=1/2$).

$\boxed{\textbf{Subcase } N=0.}$ Then $(g-1)(g+1)=0$ and we may assume $g=1$ (otherwise the transformation $x\to -x$ and $y\to y-x$ can be applied). So, $g=1$ and by translation of the origin of coordinates to the point $(-c/2,-f/2)$ we obtain the systems

$$\dot{x}=k+dy+x^2, \quad \dot{y}=l+ex+y^2. \tag{4.12}$$

For this systems we have $H_4=96(d^2+e^2)$ and

$$B_2=648\big[e^2(4k-4l-e^2)x^4+2d^2e^2(2x^2-3xy+2y^2)-d^2(4k-4l+d^2)y^4\big].$$

Therefore, the condition $B_2=0$ yields $de=e(4k-4l-e^2)=d(4k-4l+d^2)=0$ and we may assume $d=0$, since for $d\neq 0$, $e=0$ the substitution $x\leftrightarrow y$, $d\leftrightarrow e$ and $k\leftrightarrow l$ can be applied. Then the condition $B_2=0$ yields $e(4k-4l-e^2)=0$.

We claim that for $M_{\mathbf{IL}}<4$, $e\neq 0$ (i.e. $H_4\neq 0$) is necessary. Indeed, let us suppose $e=0$. Then for systems (4.12) with $d=e=0$ calculations yield:

$$\mathcal{H}=\gcd(\mathcal{E}_1,\mathcal{E}_2)=(X^2+kZ^2)(Y^2+lZ^2),$$

i.e. $\deg\mathcal{H}>3$ and our claim is proved. Thus, $H_4\neq 0$ (i.e. $e\neq 0$) and replacing e by $2e$ the condition $B_2=0$ yields $l=k-e^2$. Hence, we get the systems

$$\dot{x}=k+x^2, \quad \dot{y}=k-e^2+2ex+y^2 \tag{4.13}$$

for which calculations yield:

$$\mathcal{E}_1=-4(Y^2-eYZ+eZX+kZ^2)\mathcal{H}, \quad \mathcal{E}_2=-(X+Y-eZ)(Y^2+2eXZ-e^2Z^2+kZ^2)\mathcal{H},$$

where $\mathcal{H}=(Y-X+eZ)(X^2+kZ^2)$. Thus, $\deg\mathcal{H}=3$ and we shall show that for all values given to the parameters k and e ($e\neq 0$) the degree of $\gcd(\mathcal{E}_1,\mathcal{E}_2)$ remains 3. Indeed, since the common factor of the polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$ must depend on Y, according to Lemma 3.8 it is sufficient to observe that $\mathrm{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=-64e^2Z^2(X^2+kZ^2)^2\neq 0$ since $e\neq 0$. The systems (4.13) possess the invariant lines $x-y=e$, and $x^2+k=0$. Since for these systems $H_8=-2^93^3e^2k$ we obtain that for $k\neq 0$ $\mathrm{sign}\,(k)$=$-\mathrm{sign}\,(H_8)$.

1) If $H_8>0$ then $k<0$ and we may assume $k=-1$ via the Remark 4.3 ($\gamma=-k$, $s=1/2$). Then systems (4.13) possess the following real singular points: $(1,\pm(e-1))$ and $(-1,\pm(e+1))$, which are distinct if $e\neq\pm 1$. On the other hand for systems (4.13) with $k=-1$ we calculate $H_9=-2^{14}3^2(e^2-1)^2$ and hence we obtain Config 4.5 for $H_9\neq 0$ and Config 4.6 for $H_9=0$. In the last case we may consider $e=1$, otherwise the substitution $x\to -x$, $y\to -y$ and $t\to -t$ can be applied.

2) Assume $H_8<0$. Then $k>0$ and we may assume $k=1$ via the Remark 4.3 ($\gamma=k$, $s=1/2$). Then the invariant lines $x^2+1=0$ are imaginary and, moreover, systems (4.13) do not possess any real singular point. Thus we get Config. 4.13.

3) For $H_8=0$ since $e\neq 0$, we obtain $k=0$. We may assume $e=1$ via the Remark 4.3 ($\gamma=e$, $s=1$). This leads to the system

$$\dot{x}=x^2, \quad \dot{y}=-1+2x+y^2 \tag{4.14}$$

for which $\mathcal{H}=X^2(Y-X+Z)$. According to Lemma 3.4 the line $x=0$ could be of multiplicity two and the perturbations (4.22_ε) from Table 3 show this. Taking into consideration that both singular points $(0,\pm 1)$ of system (4.14) are double points we get Config. 4.22.

It remains to note that via the transformation $x_1=-x$ and $y_1=y-x$ the systems (4.13) can be brought to the systems

$$\dot{x}_1=-k-x_1^2, \quad \dot{y}=-e^2-2ex_1-2x_1y_1+y_1^2. \tag{4.15}$$

On the other hand we observe that systems (4.15) can be obtained from systems (4.6) by setting $g = -1$ and changing $u \to e$ and $k \to -k$. For systems (4.13) we have $H_7 = 0$ and $B_3 = -12e^2x^2(x-y)^2 \neq 0$, and for systems (4.6) we have $H_4 = 48(g-1)\big[k(g+1)^2 + 4gu^2\big]$. Since for systems (4.6) in $\mathbf{QSL_4}$ $B_3 = -3\big[k(g+1)^2 + 4gu^2\big]x^2y^2 \neq 0$ we obtain $H_4 \neq 0$. Therefore we conclude, that the representatives of the orbits in the case $N = 0$ for systems (4.15) can be included as respective particular cases of the representatives corresponding to systems (4.6) for which $N \neq 0$ as indicated in Table 2 for Config. 4.10, 4.13 and 4.22.

4.2 Systems with the divisor $D_S(C,Z) = 1\cdot\omega_1^c + 1\cdot\omega_2^c + 1\cdot\omega_3$

According to Lemma 3.7 in this case we shall consider the canonical systems $(\mathbf{S}_{II})$ for which we have:

$$\begin{array}{l} \theta = 8(h+1)[(h-1)^2+g^2], \\ N = (g^2-2h+2)x^2 + 2g(h+1)xy + (h^2-1)y^2. \end{array} \tag{4.16}$$

Remark 4.5. *We note that two infinite points of the systems $(\mathbf{S}_{II})$ are imaginary. Therefore by Lemma 3.2 we conclude that a system of this class could belong to $\mathbf{QSL_4}$ only if for this system the condition $B_3 = 0$ holds.*

In what follows we shall assume that for a system $(\mathbf{S}_{II})$ the condition $B_3 = 0$ is fulfilled.

4.2.1 The case $\theta \neq 0$

The condition $\theta \neq 0$ yields $(h+1) \neq 0$ and we may assume $c = d = 0$ via the translation $x \to x - d/(h+1)$ and $y \to y + (2dg - c(h+1))/(h+1)^2$. Thus we obtain the systems

$$\dot{x} = k + gx^2 + (h+1)xy, \quad \dot{y} = l + ex + fy - x^2 + gxy + hy^2, \tag{4.17}$$

for which we have: Coefficient$[B_3,\ y^4] = -3k(h+1)^2$. Since $h+1 \neq 0$ the condition $B_3 = 0$ implies $k = 0$ and then we obtain

$$B_3 = 3[ef(h+1) + 2gl(h-1) - f^2g]x^2(x^2-y^2) + 6[f^2 + efg - e^2h + l(h-1)^2 - g^2l]x^3y.$$

Hence, the condition $B_3 = 0$ yields the following system of equations

$$\begin{array}{l} Eq_1 \equiv ef(h+1) + 2gl(h-1) - f^2g = 0, \\ Eq_2 \equiv f^2 + efg - e^2h + l\big[(h-1)^2 - g^2\big] = 0. \end{array} \tag{4.18}$$

Because $\theta \neq 0$ the conditions $gl(h-1) = (h-1)^2 - g^2 = 0$ are impossible. Hence we calculate

$$\text{Res}_l(Eq_1,\, Eq_2) = (e - eh + fg)[2egh + f(h^2 - 1 - g^2)].$$

On the other hand for systems (4.17) with $k = 0$ we obtain: $H_7 = 4(h+1)[e(1-h) + fg]$ and we shall consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

1) If $H_7 \neq 0$ then the equality $\text{Res}_l(Eq_1,\, Eq_2) = 0$ yields $2egh = f(g^2+1-h^2)$. Since $\theta \neq 0$ from (4.16) we have $(gh)^2 + (g^2+1-h^2)^2 \neq 0$ then without loss of generality we may set: $e = (g^2+1-h^2)u$ and $f = 2ghu$ where u is a new parameter. Therefore from (4.18) we obtain

$$g(h-1)\big[l - hu^2(g^2+(h+1)^2)\big] = 0 = \big[(h-1)^2 - g^2\big]\big[l - hu^2(g^2+(h+1)^2)\big]$$

and hence, $l = hu^2[g^2+(h+1)^2]$ which leads to the systems:

$$\begin{array}{l} \dot{x} = gx^2 + (h+1)xy, \\ \dot{y} = hu^2[g^2+(h+1)^2] + u(g^2+1-h^2)x + 2ghuy - x^2 + gxy + hy^2. \end{array} \tag{4.19}$$

For these systems we have $H_7 = 4u(h+1)^2[g^2+(h-1)^2] \neq 0$. Then $u \neq 0$ and we may assume $u = 1$ via Remark 4.3 ($\gamma = u$,$s = 1$).

For systems (4.19) with $u = 1$ calculations yield:

$$\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = X[X^2 + Y^2 - 2(h+1)XZ + 2gYZ + (g^2 + (1+h)^2)Z^2].$$

Thus deg $\mathcal{H} = 3$ and for every specific system in the family (4.19) with $u = 1$ we have $M_{\mathbf{IL}} = 4$. Since $\theta \neq 0$, by Lemma 4.2 $M_{\mathbf{IL}}$ cannot be 5. In this case systems (4.19) possess three invariant lines: $x = 0$, $y \pm i\big[x - (h+1)\big] + g = 0$ and four singularities:

$$\big(0,\, -g \pm i(h+1)\big), \quad (h+1,\, -g), \quad \big(-h(h+1),\, gh\big).$$

Due to the condition $\theta \neq 0$ (i.e. $h+1 \neq 0$) all points are distinct and the intersection point $(h+1,\, -g)$ of the imaginary lines is not placed on the line $x = 0$. This leads to the Config. 4.2.

2) Assume $H_7 = 0$. Then $e(h-1) = fg$ and since the condition $\theta \neq 0$ yields $g^2 + (h-1)^2 \neq 0$, we may assume $e = gu$ and $f = (h-1)u$, where u is a new parameter. Then from (4.18) we have

$$g(h-1)(l+u^2) = 0 = [(h-1)^2 - g^2](l+u^2).$$

By $g^2 + (h-1)^2 \neq 0$ evidently we obtain $l = -u^2$ and this leads to the systems:

$$\dot{x} = gx^2 + (h+1)xy, \quad \dot{y} = -u^2 + gu\,x + u(h-1)y - x^2 + gxy + hy^2. \tag{4.20}$$

For these systems we have: $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2X\big[X^2 + (Y+uZ)^2\big]$. Then for a specific systems in this family $M_{\mathbf{IL}} \geq 4$, and by Lemma 4.2 due to the condition $\theta \neq 0$, $M_{\mathbf{IL}}$ cannot be 5. The systems (4.20) possess three invariant lines: $x = 0$ and $x \pm i(y+u) = 0$ and the following singular points:

$$(0, -u), \quad (0, u/h), \quad \Big(\frac{u(h+1)}{g \pm i(h+1)},\, -\frac{gu}{g \pm i(h+1)}\Big)$$

We observe that all three lines have the common point of intersection: $(0, -u)$. Moreover if $u = 0$ this point becomes of multiplicity 4 and the point $(0, u/h)$ tends to infinity when h tends to zero. On the other hand for systems (4.20) we have $\mu = -32h\big[g^2 + (h+1)^2\big]$ and $H_9 = 2^8 3^2 u^8 (h+1)^8$. Thus in the case $H_7 = 0$ we obtain: Config. 4.6 for $\mu H_9 \neq 0$, Config. 4.8 for $\mu \neq 0$, $H_9 = 0$ and Config. 4.7 for $\mu = 0$ (in this case $u \neq 0$ otherwise we get degenerate systems). It remains to note that if $u \neq 0$ we may assume $u = 1$ via Remark 4.3 ($\gamma = u$, $s = 1$).

4.2.2 The case $\theta = 0$

According to (4.16) we have $(h+1)[(h-1)^2 + g^2] = 0$ and we shall consider two subcases $N \neq 0$ and $N = 0$.

Subcase $N \neq 0$. Then by (4.16) the condition $\theta = 0$ yields $h = -1$ and in addition we may assume $f = 0$ due to the translation: $x \to x$ and $y \to y + f/2$. Hence, we obtain the systems

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + ex - x^2 + gxy - y^2, \tag{4.21}$$

for which by Remark 4.5 the condition $B_3 = 0$ must be satisfied. Calculation yields

$$\text{Coefficient}[B_3, y^4] = -3\,d^2 g, \quad H_7 = 4d(g^2 + 4).$$

So the condition $B_3 = 0$ implies $dg = 0$. We claim that for systems (4.21) to be in the class $\mathbf{QSL_4}$ the condition $d \neq 0$ must be fulfilled. Indeed, suppose $d = 0$. Then we have:

$$B_3 = 3(2ce - 4gl + 4k - g^2k)x^2(x^2 - y^2) - 6(c^2 - e^2 - 4l + lg^2 - 4gk)x^3y$$

and hence the condition $B_3 = 0$ yields a system of two linear equations with respect to the parameters k and l. As its determinant equals $(g^2+4)^2 \neq 0$ we obtain the unique solution:

$$\begin{aligned} k &= 2(2c+eg)(cg-2e)/(g^2+4)^2, \\ l &= (4c^2-4e^2+8ceg-c^2g^2+e^2g^2)/(g^2+4)^2. \end{aligned} \tag{4.22}$$

Then for systems (4.21) with these values of the parameters l and k and with $d = 0$ calculations yield:

$$\begin{aligned} \mathcal{H} = 2\Big[(4+g^2)X - (2e-cg)Z\Big]\Big[g(g^2+4)X + 2(2c+eg)Z\Big] \times \\ \Big[(4+g^2)(X^2+Y^2) - 2(2e-cg)XZ + 2(2c+eg)YZ + (c^2+e^2)Z^2\Big], \end{aligned}$$

i.e. in this case the family of systems (4.21) possess four affine lines, which implies $M_{\mathbf{IL}} \geq 5$. Our claim is proved.

Let us assume $d \neq 0$, i.e. $H_7 \neq 0$. Then $g = 0$ and we may assume $e = 0$ via the translation: $x \to x + e/2$, $y \to y$. After that for systems (4.21) calculations yield:

$$B_3 = 12kx^2(x^2-y^2) - 6(c^2-4l+d^2)x^3y.$$

Therefore the condition $B_3 = 0$ yields $k = 0$ and $l = (c^2+d^2)/4$. Then by replacing c with $2c$ and d with $2d$ and assuming $d = 1$ (due to the Remark 4.3 ($\gamma = d$, $s = 1$) we get the systems:

$$\dot{x} = 2cx + 2y, \quad \dot{y} = c^2 + 1 - x^2 - y^2, \tag{4.23}$$

for which we calculate

$$\begin{aligned} \mathcal{E}_1 &= 2\big[X^2 - 2cXY - Y^2 + (c^2+1)(2X+Z)Z\big]\mathcal{H}, \\ \mathcal{E}_2 &= (cX+Y)[X^2+Y^2+2XZ-2cYZ+(c^2+1)Z^2]\mathcal{H}, \\ \mathcal{H} &= \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2Z[(X-Z)^2 + (Y+cZ)^2]. \end{aligned}$$

Thus, for this family of systems we obtain $\deg \mathcal{H} = 3$. According to Lemma 3.8 to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ it is necessary that the following holds in $\mathbb{R}[X, Z]$:

$$\mathrm{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 32(c^2+1)^2(X+dZ)^6 = 0.$$

However this is impossible and hence the systems (4.23) belong to the class $\mathbf{QSL_4}$ for any value of the parameter c. Note that these systems possess the invariant lines $y + c = \pm i(x-1)$. Since $Z \mid \mathcal{H}$, by Corollary 3.7 we conclude that the line $Z = 0$ could be of multiplicity two and this is confirmed by the perturbations $(IV.27_\varepsilon)$ from Table 3.

The systems (4.11) possess two finite distinct singular points: $(-1, c)$ and $(1, -c)$ and the second one is the point of intersection of the imaginary lines. This leads to the Config. 4.27.

Subcase $N = 0$. Then from (4.16) we have $g = h - 1 = 0$ and without loss of generality we may assume $c = d = 0$ via the translation $x \to x - d/2$, $y \to y - c/2$. Hence we obtain the systems

$$\dot{x} = k + 2xy, \quad \dot{y} = l + ex + fy - x^2 - y^2,$$

for which calculations yield:

$$B_3 = 6\ \big[(ef-2k)x^4 + (f^2-e^2)x^3y - (4k+ef)x^2y^2 - 2ky^4\big].$$

Therefore, the condition $B_3 = 0$ yields $k = e = f = 0$ and this leads to the following systems

$$\dot{x} = 2xy, \qquad \dot{y} = l - x^2 + y^2,$$

for which we obtain: $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2X\left[X^4 + 2X^2(Y^2 - lZ^2) + (Y^2 + lZ^2)^2\right]$. Hence, for $N = 0 = B_3$ we have $\deg \mathcal{H} = 5$ and systems $(\mathbf{S}_{II})$ cannot belong to the class $\mathbf{QSL_4}$.

4.3 Systems with $D_S(C, Z) = 2 \cdot \omega_1 + 1 \cdot \omega_2$

For this case we shall later need the following T-comitant.

Notation 4.6. *Let us denote* $H_{11}(a, x, y) = 8H\big[(C_2, D)^{(2)} + (D, D_2)^{(1)}\big] + 3H_2^2.$

By Lemma 3.7 systems (2.1) can be brought via a linear transformation to the canonical form $(\mathbf{S}_{III})$ for which we have:

$$\theta = -8h^2(g-1), \quad \mu = 32gh^2, \quad N = (g^2-1)x^2 + 2h(g-1)xy + h^2y^2. \tag{4.24}$$

4.3.1 The case $\theta \neq 0, B_3 = 0$

Then $h(g-1) \neq 0$ and we may assume $h = 1$ due to the substitution $y \to y/h$. Then via the translation $x \to x - d$ and $y \to y + 2dg - c$ we may assume $c = d = 0$. Thus we obtain the systems

$$\dot{x} = k + gx^2 + xy, \quad \dot{y} = l + ex + fy + (g-1)xy + y^2, \tag{4.25}$$

for which calculation yields

$$B_3 = -3\big[l(g-1)^2 + ef(1-g) + e^2\big]x^4 + 3(l - 2k - 2gk)x^2y^2 - 6kxy^3.$$

Hence the condition $B_3 = 0$ implies $k = l = 0$ and $e(f - fg + e) = 0$. On the other hand for the systems (4.25) we have $H_7 = -4(f - fg + e)$ and we shall consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

Subcase $H_7 \neq 0$. Then $f - fg + e \neq 0$ and the condition $B_3 = 0$ yields $e = 0$. Therefore the condition $H_7 = 4f(g-1) \neq 0$ implies $f \neq 0$ and we may assume $f = 1$ via Remark 4.3 ($\gamma = f$, $s = 1$). This leads to the systems:

$$\dot{x} = gx^2 + xy, \quad \dot{y} = y + (g-1)xy + y^2, \tag{4.26}$$

for which we calculate:

$$\begin{aligned}
&\mathcal{E}_1 = 2\big[g(g-1)X^2 + 2gXY + Y^2 + 2gXZ - gZ^2\big]\mathcal{H},\\
&\mathcal{E}_2 = (gX+Y)^2(gX - X + Y + Z)\mathcal{H}, \quad \mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = X^2Y.
\end{aligned}$$

Thus, we obtain $\deg \mathcal{H} = 3$ and we claim that for systems (4.26) we cannot have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_1$ for any choice of value for $g \notin \{0, 1\}$. Indeed, by Lemma 3.8 to have such a common factor it is necessary that the following holds in $\mathbb{R}[X, Z]$:

$$\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8(1-g)g^2(X-Z)^6 = 0.$$

As $g \notin \{0, 1\}$ this is impossible.

We observe that the systems (4.26) possess the invariant affine lines $x = 0$ and $y = 0$ and the line $x = 0$ is a double one. This is confirmed by the perturbations $(IV.25_\varepsilon)$ from Table 3.

Note that the systems (4.26) have three distinct finite singular points: $(0, 0)$ (which is double), $(0, -1)$ and $(1, -g)$. Therefore we obtain Config. 4.25.

Subcase $H_7 = 0$. Then $e = f(g-1)$ and we obtain the systems:

$$\dot{x} = g\,x^2 + xy, \quad \dot{y} = f(g-1)\,x + f\,y + (g-1)\,xy + y^2, \tag{4.27}$$

for which calculations yield:

$$\begin{aligned}
&\mathcal{E}_1 = 2\big[g(g-1)X^2 + 2gXY + Y^2 + f(1-g)XZ - fYZ\big]\mathcal{H}, \qquad \mathcal{H} = X^2(Y + fZ),\\
&\mathcal{E}_2 = 3(gX+Y)^2(gX - X + Y)\mathcal{H}, \quad \operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8(1-g)(gX - fZ)^2X^4.
\end{aligned}$$

Thus, we obtain deg $\mathcal{H}=3$ and since $\text{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})\not\equiv 0$, according to Lemma 3.8 we cannot have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_1$ for any specific value of the parameters.

Note that the systems (4.27) have the finite singular points: $(0,0)$ (which is double), $(0,-f)$ and $(f/g,-f)$. For $f\neq 0$ (then we can assume $f=1$ due to the Remark 4.3 ($\gamma=f$, $s=1$)) these points are distinct and finite if $g\neq 0$. When $g\to 0$ the point $(f/g,-f)$ tends to infinity. For $f=0$ (then $g\neq 0$, otherwise we get a degenerate system) the point $(0,0)$ is of multiplicity four.

On the other hand for the systems (4.27) we have: $D=-fx^2y$ and $\mu=32g$, i.e. the conditions $f=0$ and $g=0$ are captured by the T-comitants D and μ, respectively.

We observe that the systems (4.27) possess the invariant affine lines $x=0$ and $y+f=0$ and the line $x=0$ is a double one. This is confirmed in the case $f\neq 0$ (respectively $f=0$) by the perturbations ($IV.21_\varepsilon$) and ($IV.26_\varepsilon$) (respectively ($IV.20_\varepsilon$)) from Table 3.

Thus for $\theta\neq 0$, $B_3=0$ and $H_7=0$, if $D\neq 0$ and $\mu\neq 0$ (respectively $D\neq 0$ and $\mu=0$; $D=0$) we obtain Config. 4.21 (respectively Config. 4.26; Config. 4.20).

4.3.2 The case $\theta=0=B_2$

Subcase $\mu\neq 0$. From (4.24) we obtain $h\neq 0$, $g=1$ and then we may assume $h=1$ due to the change $y\to y/h$. Moreover, we may assume $c=d=0$ via the translation $x\to x-d$ and $y\to y+2d-c$. So, we obtain the canonical systems

$$\dot{x}=k+x^2+xy,\quad \dot{y}=l+ex+fy+y^2, \tag{4.28}$$

for which calculation yields:

$$B_2=-648\Big[e^2(l-4k)x^4-4e^2kx^3y+k^2y^4\Big],\quad H_7=-4e.$$

Hence, the condition $B_2=0$ yields $k=el=0$. We claim, that if $e\neq 0$ (i.e. $H_7\neq 0$) then no system of the family (4.28) can belong to $\mathbf{QSL_4}$. Indeed, supposing $e\neq 0$ (then we may assume $e=1$ due to the Remark 4.3 ($\gamma=e$, $s=1$)), we have $l=0$ and we obtain the systems

$$\dot{x}=x^2+xy,\quad \dot{y}=x+fy+y^2$$

for which we calculate:

$$\begin{aligned}\mathcal{E}_1&=2[X^2Z+2XY^2+Y^3+(2f-1)XYZ+(1-f)XZ^2+f(1-f)YZ^2]\mathcal{H},\\ \mathcal{E}_2&=(X+Y)^2(Y^2+XZ+fYZ)\mathcal{H},\quad \mathcal{H}=X^2.\end{aligned}$$

So, deg $\mathcal{H}=2$ and according to Lemma 3.8 to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ for some specific value of f the condition $\text{Res}_Y(\mathcal{E}_1/\mathcal{F},\ \mathcal{E}_2/\mathcal{F})=16X^4Z^2(X-fZ+Z)^6=0$ must hold in $\mathbb{R}[X,Z]$ and this is impossible.

Assuming $e=0$ (then $H_7=0$) we get the following systems:

$$\dot{x}=x^2+xy,\quad \dot{y}=l+fy+y^2 \tag{4.29}$$

for which calculations yield:

$$\mathcal{E}_1=2(2X^2+XY-fXZ+lZ^2)\mathcal{H},\quad \mathcal{E}_2=X(X+Y)^2\mathcal{H},$$

where $\mathcal{H}=X(Y^2+fYZ+lZ^2)$. Thus, we obtain deg $\mathcal{H}=3$ and to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ for some specific value of (f,l), according to Lemma 3.8 it is necessary and sufficient that the following condition holds:

$$\text{Res}_X(\mathcal{E}_1/\mathcal{F},\ \mathcal{E}_2/\mathcal{F})=8lZ^2(Y^2+fYZ+lZ^2)^2\equiv 0,$$

which is equivalent to $l=0$. On the other hand for the systems (4.29) we have $B_3 = 3lx^2y^2$ and hence for these systems to belong to $\mathbf{QSL_4}$ we must have $B_3 \neq 0$.

Assume $l \neq 0$ (i.e. $B_3 \neq 0$). Then the systems (4.29) possess three invariant lines: $x=0$ and $y^2+fy+l=0$ and the position of these two lines depends on the values of the affine invariant $H_{10} = 8(f^2-4l)$.

1) Assume $H_{10} > 0$. Then we may set $f^2-4l = 4u^2 \neq 0$ (u is a new parameter) and replacing f by $2f$ we have $l = f^2-u^2$. Assuming $u=1$ via Remark 4.3 ($\gamma = u$, $s=1$) we get the systems

$$\dot{x} = x^2+xy, \quad \dot{y} = (y+f)^2-1. \tag{4.30}$$

These systems possess three real distinct invariant lines ($x=0$ and $y+f=\pm 1$) and the following singular points:

$$(0, 1-f), \quad (0,-1-f), \quad (-1+f, 1-f), \quad (1+f, -1-f).$$

Since for systems (4.30) we have $B_3 = 3(f^2-1)x^2y^2) \neq 0$, we obtain $|f| \neq 1$ and, hence, all singular points are distinct. Thus we get Config. 4.11.

2) If $H_{10} < 0$ then as above assuming $f^2-4l = -4u^2 \neq 0$ and replacing f by $2f$ and $u=1$ we get the systems

$$\dot{x} = x^2+xy, \quad \dot{y} = (y+f)^2+1 \tag{4.31}$$

which possess one real ($x=0$) and two imaginary ($y+f=\pm i$) invariant lines. Since all singular points are imaginary we obtain Config 4.23.

3) Assume now $H_{10}=0$, i.e. $f^2=4l$. Then replacing f by $2f$ we obtain the systems

$$\dot{x} = x^2+xy, \quad \dot{y} = (y+f)^2 \tag{4.32}$$

for which we have $B_3 = 3f^2x^2y^2 \neq 0$. Then $f \neq 0$ and we may assume $f=1$ due to the Remark 4.3 ($\gamma = f$, $s=1$). We observe that the system (4.32) has a simple invariant line ($x=0$) and a double one ($y=-1$). Moreover, these systems have 2 double singular points: $(0,-1)$ and $(1,-1)$. Thus we obtain Config. 4.23.

Subcase $\mu = 0$. Since $\theta = 0$ from (4.24) we obtain $h=0$ and for the systems $(\mathbf{S}_{III})$ we calculate

$$N = (g^2-1)x^2, \quad H_7 = 4d(g^2-1).$$

I. If $N \neq 0$ then $g-1 \neq 0$ and we may assume $e=f=0$ via the translation $x \to x+f/(1-g)$ and $y \to y+e/(1-g)$. This leads to the systems

$$\dot{x} = k+cx+dy+gx^2, \quad \dot{y} = l+(g-1)xy, \tag{4.33}$$

for which calculations yield:

$$B_2 = 648d\Big[cl(g-1)^3x^4 + 4dlg(g-1)^2x^3y - d^3g^2y^4\Big]$$

and we shall consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

1) Suppose $H_7 \neq 0$. Then $d \neq 0$ and we may assume $d=1$ due to the substitution $y \to y/d$. Then by $g-1 \neq 0$ the condition $B_2 = 0$ yields $g = cl = 0$.

We claim that the systems (4.33) with $d=1$ could be in the class $\mathbf{QSL_4}$ only if the condition $c=0$ is fulfilled. Indeed, supposing $c \neq 0$ from $B_2=0$ we obtain $g=l=0$ and this leads to the systems

$$\dot{x} = k+cx+y, \quad \dot{y} = -xy.$$

For these systems calculations yield:

$$\mathcal{E}_1 = -2(cX^3 + kX^2Z - cXYZ - Y^2Z - ckXZ^2 - 2kYZ^2 - k^2Z^3)\mathcal{H},$$
$$\mathcal{E}_2 = -XZ(cX + Y + kZ)^2\mathcal{H}, \quad \mathcal{H} = YZ.$$

So, $\deg \mathcal{H} = 2$ and since $c \neq 0$, to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ for some specific value of (c, k), according to Lemma 3.8 it is necessary that the following holds in $\mathbb{R}[Y, Z]$:

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -8c^2Y^2Z^4(Y + kZ)^6 = 0,$$

which is impossible since $c \neq 0$.

Let us now assume $c = 0$. Then we obtain the systems:

$$\dot{x} = k + y, \quad \dot{y} = l - xy,$$

for which we calculate:

$$\mathcal{E}_1 = 2(-kX^2Y + Y^3 - lXYZ + 2kY^2Z + klXZ^2 + k^2YZ^2 + l^2Z^3)\mathcal{H}, \quad \mathcal{H} = Z^2,$$
$$\mathcal{E}_2 = -(Y + kZ)^2(XY - lZ^2)\mathcal{H}, \quad \operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -8lZ^6(kX + lZ)^6.$$

Thus, we have $\deg \mathcal{H} = 2$ and to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ for some specific value of (k, l), according to Lemma 3.8 it is necessary $\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) \equiv 0$. This condition is equivalent to $l = 0$ in which case the additional common factor is Y. Thus the systems (4.33) with $d = 1$ and $g = cl = 0$ belong to the class $\mathbf{QSL_4}$ if and only if $c = l = 0$. Since for these systems we have $B_3 = -3x^2(lx^2 - cy^2)$ we conclude that the condition $c = l = 0$ is equivalent to $B_3 = 0$.

Therefore, for $\theta = \mu = 0$, $N \neq 0$, $H_7 \neq 0$ and $B_3 = 0$ (then $B_2 = 0$) we obtain the systems:

$$\dot{x} = k + y, \quad \dot{y} = -xy, \tag{4.34}$$

for which we have $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = YZ^2$ and

$$\mathcal{E}_1 = (-kX^2 + d^2Y^2 + 2dkYZ + k^2Z^2)\mathcal{H}, \quad \mathcal{E}_2 = -3X(dY + kZ)^2\mathcal{H}.$$

Hence $\deg \mathcal{H} = 3$ and according to Lemma 3.8 to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ for some specific value of k it is necessary to have $\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8k^2X^6 \equiv 0$, i.e. $k = 0$. This case is ruled out as the corresponding system (4.34) is degenerate.

The systems (4.34) possess the invariant line $y = 0$ and due to Corollary 3.7 since $Z^2 \mid \mathcal{H}$ the line $Z = 0$ could be of the multiplicity three. And this is confirmed by the perturbations $(IV.35_\varepsilon)$ from Table 3. So in this case we obtain Config. 4.35.

2) Assume $H_7 = 0$. Then $d = 0$ and this implies $B_2 = 0$. Hence we get the systems

$$\dot{x} = k + cx + gx^2, \quad \dot{y} = l + (g - 1)xy, \tag{4.35}$$

for which calculations yield:

$$\mathcal{E}_1 = 2\big[(g - 1)X^2Y + l(g + 1)XZ^2 + k(1 - g)YZ^2 + clZ^3\big]\mathcal{H},$$
$$\mathcal{E}_2 = (gX^2 + cXZ + kZ^2)(-XY + gXY + lZ^2)\mathcal{H}, \quad \mathcal{H} = gX^2 + cXZ + kZ^2,$$

Thus, we have $\deg \mathcal{H} = 2$, but for systems (4.35) to be in the class $\mathbf{QSL_4}$ we need an additional common factor of of $\mathcal{E}_1$ and $\mathcal{E}_2$. Since $g - 1 \neq 0$ according to Lemma 3.8 we can obtain such a common factor if at least one of the following two identities holds:

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{F},\ \mathcal{E}_2/\mathcal{F}) = 8(g-1)\Big[c^2 - k(g + 1)^2\Big]\Big[k(g - 1)^2Y^2 + c\,l(1-g)YZ + g\,l^2Z^2\Big]^2 YZ^6 \equiv 0,$$
$$\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{F},\ \mathcal{E}_2/\mathcal{F}) = 2\,l(g - 1)(gX^2 + cXZ + kZ^2)^2Z^2 \equiv 0.$$

So to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ one of the following three conditions must be fulfilled:

$$(i)\ \ l=0; \qquad (ii)\ \ c^2-k(g+1)^2=0; \qquad (iii)\ \ k=c\,l=g\,l=0.$$

Since for $l\neq 0$ the condition (iii) leads to the degenerate systems (4.35), only the conditions (i) and (ii) remain to be examine.

On the other hand, for the systems (4.35) we have

$$B_3=-3\,l(g-1)^2x^4, \qquad H_6=128(g-1)^4[k(g+1)^2-c^2]x^6$$

and we shall consider two subcases: $B_3=0$ and $B_3\neq 0$, $H_6=0$.

a) If $B_3=0$ (i.e. $l=0$) we get the systems:

$$\dot{x}=k+cx+gx^2, \quad \dot{y}=(g-1)xy \tag{4.36}$$

with $k\neq 0$, otherwise they are degenerate. For these systems calculation yields:

$$\begin{aligned}&\mathcal{E}_1=2(X^2-kZ^2)\mathcal{H}, \qquad \mathcal{H}=\gcd(\mathcal{E}_1,\mathcal{E}_2)=(g-1)(gX^2+cXZ+kZ^2)Y,\\ &\mathcal{E}_2=X(gX^2+cXZ+kZ^2)\mathcal{H}, \qquad \operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=8k^2[c^2-k(g+1)^2]Z^6.\end{aligned}$$

Thus, we have $\deg\mathcal{H}=3$ and since $k\neq 0$ according to Lemma 3.8 if $c^2-k(g+1)^2\neq 0$ we cannot have an additional common factor of of $\mathcal{E}_1$ and $\mathcal{E}_2$. The last condition is equivalent to $H_6\neq 0$.

We observe, that the systems (4.36) have the invariant lines $y=0$ and $gx^2+cx+k=0$. The positions of the last two lines depend on the values of the T-comitants H_{11} and K, because for these systems we have

$$H_{11}=48(g-1)^4(c^2-4gk)x^4, \quad K=2g(g-1)x^2$$

a.1) Assume $K\neq 0$. In this case $g\neq 0$ and both invariant lines $gx^2+cx+k=0$ are affine.

α) If $H_{11}>0$ then without loss of generality we may introduce two new parameters u and v as follows: $c^2-4gk=4g^2u^2>0$ and $c=2gv$. Then $k=g(v^2-u^2)$ and and since $u\neq 0$ we may assume $u=1$ via Remark 4.3 ($\gamma=u$, $s=1$). Then we obtain the systems

$$\dot{x}=g\big[(x+v)^2-1\big], \quad \dot{y}=(g-1)xy, \tag{4.37}$$

which possess three real distinct invariant lines ($y=0$ and $x+v=\pm 1$) and two finite singular points: $(-v\pm 1,\,0)$. We note that $v\neq\pm 1$, otherwise we get degenerate systems (4.37). Thus for $K\neq 0$ and $H_{11}>0$ we obtain Config. 4.12.

β) For $H_{11}<0$ as above we may set $c^2-4gk=-4g^2u^2<0$ and $c=2gv$. Then $k=g(v^2+u^2)$ and assuming $u=1$ due to Remark 4.3 ($\gamma=u$, $s=1$) we obtain the systems

$$\dot{x}=g\big[(x+v)^2+1\big], \quad \dot{y}=(g-1)xy$$

with invariant lines $y=0$ and $x+v=\pm i$. These systems have no real singularities and we get Config. 4.15.

γ) Assume now that $H_{11}=0$, i.e. $c^2=4gk$. Setting $c=2gv$ we obtain $k=gv^2$ and this leads to the systems

$$\dot{x}=g(x+v)^2, \quad \dot{y}=(g-1)xy.$$

We observe that the line $x=-v$ is of multiplicity two (it cannot be of multiplicity three in view of Lemma 3.4, as $\mathcal{H}=Y(X+vZ)^2$). Since $v\neq 0$ (otherwise systems become degenerate) we may assume $v=1$ via Remark 4.3 ($\gamma=v$, $s=1$) and in this case we obtain Config. 4.24.

a.2) Let us consider the case $K = 0$. Then $g = 0$ and the systems (4.36) become

$$\dot{x} = k + cx, \quad \dot{y} = -xy \tag{4.38}$$

for which $\mathcal{H} = YZ(cX + kZ)$. Since in this case $H_{11} = 48c^2x^4$ we shall consider two cases: $H_{11} \neq 0$ and $H_{11} = 0$.

α) If $H_{11} \neq 0$ then $c \neq 0$ and we may assume $c = 1$ via Remark 4.3 ($\gamma = c$, $s = 1$). So the systems (4.38) with $c = 1$ possess invariant affine lines $y = 0$ and $x = -k$. Moreover, the line $Z = 0$ is double as it is confirmed by the perturbations ($IV.19_\varepsilon$) from Table 3. Taking into account the existence of a unique finite singular point $(-k, 0)$ we obtain Config. 4.19.

β) For $H_{11} = 0$ we have $c = 0$ and then we may assume $k \in \{-1, 1\}$ via Remark 4.3 ($\gamma = |k|$, $s = 1/2$). In these cases the systems (4.38) with $c = 0$ possess only one invariant affine line ($y = 0$), and the line $Z = 0$ is triple, as it is confirmed by the perturbations ($IV.36_\varepsilon$) from Table 3. Therefore we get Config. 4.36.

b) Assume now $B_3 \neq 0$ and $H_6 = 0$. Then $l \neq 0$ and the condition $c^2 - k(g+1)^2 = 0$ is fulfilled. Since $g + 1 \neq 0$ (due to $N \neq 0$) we may use a new parameter u: $c = u(g+1)$ and then $k = u^2$. This leads to the systems

$$\dot{x} = (u + x)(u + gx), \quad \dot{y} = l + (g-1)xy, \tag{4.39}$$

for which calculation yields:

$$\begin{aligned}
&\mathcal{E}_1 = 2\big[(g-1)XY + u(1-g)YZ + l(g+1)Z^2\big]\mathcal{H},\\
&\mathcal{E}_2 = (gX + cuZ)(-XY + gXY + lZ^2)\mathcal{H},\\
&\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = (X + uZ)^2(gX + uZ),\\
&\text{Res}_Z(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8\,l\,(g-1)^2(g+1)\big[lg^2X + u^2(g-1)Y\big]^2X^2Y^2.
\end{aligned}$$

So, we have $\deg \mathcal{H} = 3$ and according to Lemma 3.8 to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ for some specific value of (g, l, u) we must have $\text{Res}_Z(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 0$ in $\mathbb{R}[X, Y]$. Therefore, since $l(g^2 - 1) \neq 0$, we obtain $g = u = 0$, however this condition leads to degenerate systems (4.39). Hence for $l(g^2 - 1) \neq 0$ (i.e. $B_3N \neq 0$) the systems (4.39) belong to the class $\mathbf{QSL_4}$.

We observe that the systems (4.39) possess invariant lines $x + u = 0$ and $gx + u = 0$ and by Lemma 3.4 and the expression for $\mathcal{H}$ the first line could be a double one. Clearly, the values of the parameters g and u govern the position of the invariant lines in the configuration. On the other hand for systems (4.39) we have:

$$K = 2g(g-1)x^2, \quad H_{11} = 48u^2(g-1)^6x^4.$$

b.1) Assume $KH_{11} \neq 0$. Then $gu \neq 0$ and we may set $u = 1$ via Remark 4.3 ($\gamma = u$, $s = 1$). Moreover due to the additional change $y \to ly$ we may also assume $l = 1$. In this case the systems (4.39) possess two parallel invariant affine lines $x = -1$ and $gx = -1$. The first line is a double one as it is confirmed by the perturbations ($IV.30_\varepsilon$) from Table 3. Taking into account the singular points $\big(-1,\ 1/(g-1)\big)$ and $\big(g/(g-1),\ -1/g\big)$ which are finite and distinct since $g(g-1) \neq 0$, we get Config. 4.30.

b.2) For $K \neq 0$ and $H_{11} = 0$ we have $g \neq 0$, $u = 0$ and the systems (4.39) possess the invariant line $x = 0$ which is of the multiplicity three. This is confirmed by the perturbations ($IV.43_\varepsilon$) from Table 3. Since $l \neq 0$ (then we may assume $l = 1$ via the change $y \to ly$) these systems do not have finite singular points. So we obtain Config. 4.43.

b.3) Finally, assume that $K = 0$, i.e. $g = 0$. Then $u \neq 0$ (otherwise we get degenerate systems) and as above we may assume $u = 1$ and $l = 1$. So we obtain the system

$$\dot{x} = 1 + x, \quad \dot{y} = 1 - xy, \tag{4.40}$$

for which $\mathcal{H} = Z(X+Z)^2$, i.e. the line $x+1=0$ as well as the line $Z=0$ are of the multiplicity two. This is confirmed by the perturbations $(IV.40_\varepsilon)$ from Table 3. Considering the existence of a simple finite singular point $(-1,-1)$ we obtain Config. 4.40.

$\boxed{\textbf{II.}}$ Assume $N=0$. According to (4.24) the condition $\theta = \mu = 0$ yields $h=0$ for systems $(\mathbf{S}_{III})$ and then for these systems we have:

$$N = (g^2-1)x^2, \qquad K = 2g(g-1)x^2.$$

So, the condition $N=0$ implies either $g=1$ or $g=-1$ and we shall consider two subcases: $K \neq 0$ and $K=0$.

1) For $K \neq 0$ we obtain $g=-1$ and we may assume $e=f=0$ via the translation $x \to x+f/2$ and $y \to y+e/2$. Then calculations yield:

$$B_2 = -648\, d(8clx^4 + 16dlx^3y + d^3y^4).$$

Hence, the condition $B_2 = 0$ implies $d=0$ and we obtain the systems:

$$\dot{x} = k + cx - x^2, \quad \dot{y} = l - 2xy. \tag{4.41}$$

for which calculations yield:

$$\mathcal{E}_1 = 2(-2X^2Y + 2kYZ^2 + clZ^3)\mathcal{H}, \quad \mathcal{E}_2 = (X^2 - cXZ - kZ^2)(2XY - lZ^2)\mathcal{H},$$

where $\mathcal{H} = -X^2 + cXZ + kZ^2$. Thus, we have $\deg \mathcal{H} = 2$ and we observe that an additional common factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ must depend on X or/and on Y. According to Lemma 3.8 this occurs if and only if at least one of the following two identities holds:

$$\begin{aligned}
\mathrm{Res}\,_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) &= -16c^2YZ^6(4kY^2 + 2c\,lYZ - l^2Z^2)^2 \equiv 0,\\
\mathrm{Res}\,_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) &= -4\,lZ^2(-X^2 + cXZ + kZ^2)^2 \equiv 0.
\end{aligned}$$

Since $k=l=0$ yield degenerate systems we obtain the condition $cl=0$.

We claim that for the systems (4.41) to be in the class $\mathbf{QSL_4}$ it is necessary to have $c \neq 0$ and $l=0$. Indeed, supposing that $c=0$ we get the systems

$$\dot{x} = k - x^2, \quad \dot{y} = l - 2xy,$$

for which calculations yield: $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2(X^2 - kZ)^2$. Hence, $\deg \mathcal{H} = 4$, i.e. these systems belong to the class $\mathbf{QSL_5}$ and not to $\mathbf{QSL_4}$.

Since for systems (4.41) we have $B_3 = -12lx^4$ and $H_6 = -2^{11}c^2x^6$, in what follows we shall consider $B_3 = 0$ and $H_6 \neq 0$ (i.e. $l=0$ and $c \neq 0$). In this case we get the systems

$$\dot{x} = k + cx - x^2, \quad \dot{y} = -2xy, \tag{4.42}$$

which in fact are particular case of systems (4.36) (see page 31). More exactly, when $g=-1$ from (4.36) we obtain (4.42). On the other hand when $g=-1$ for systems (4.36) we have

$$H_{11} = 768(c^2+4k)x^4, \quad K = 4x^2, \quad H_6 = -2^{11}c^2x^6,$$

i.e. $H_6K \neq 0$. Then as it was proved on the page 31 if $H_{11} > 0$ (respectively $H_{11} < 0$; $H_{11} = 0$) we obtain Config. 4.12 (respectively Config. 4.15; Config. 4.24).

2) For $K=0$ and $N=0$ we obtain $g=1$ and via the translation $x \to x - c/2$ and $y \to y$ the systems $(\mathbf{S}_{III})$ can be brought to the systems

$$\dot{x} = k + dy + x^2, \quad \dot{y} = l + ex + fy,$$

for which

$$B_2 = -648d^4y^4, \quad B_3 = 6d(fx - dy)xy^2.$$

Hence, the condition $B_2 = 0$ yields $d = 0$ which implies $B_3 = 0$ and we obtain the systems:

$$\dot{x} = k + x^2, \quad \dot{y} = l + ex + fy. \tag{4.43}$$

Calculations yield:

$$\begin{aligned}
&\mathcal{E}_1 = 2\big[eX^2 + 2fXY + (2\,l - ef)XZ - f^2YZ - (ek + fl)Z^2\big]\mathcal{H},\\
&\mathcal{E}_2 = (eX + fY + lZ)(X^2 + kZ^2)\mathcal{H}, \quad \mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = Z(X^2 + kZ^2).
\end{aligned}$$

So, we have deg $\mathcal{H} = 3$ and

$$\begin{aligned}
&\mathrm{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -8e(f^2 + 4k)Z^2\Big[f^2Y^2 + 2flYZ + (e^2k + l^2)Z^2\Big]^2,\\
&\mathrm{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -2ef(X^2 + k^2Z^2),\\
&\mathrm{Res}_Z(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -8\,e(f^2 + 4k)\Big[(e^2k + l^2)X^2 + 2efkXY + f^2kY^2\Big]^2X^2.
\end{aligned}$$

We note, that the conditions $f = e^2k + l^2 = 0$ yield degenerate systems (4.43), and for $f = 0$ both polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$ do not depend on Y. Hence, according to Lemma 3.8, to have an additional common factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ the condition $e(f^2 + 4k) = 0$ has to be satisfied. Therefore the systems (4.43) will belong to $\mathbf{QSL_4}$ only if $e(f^2 + 4k) \neq 0$. On the other hand, for systems (4.43) we have:

$$N_1 = 8ex^4, \quad N_2 = 4(f^2 + 4k)x, \quad N_5 = -64kx^2$$

and hence, the condition $e(f^2 + 4k) \neq 0$ is equivalent to $N_1N_2 \neq 0$.

The systems (4.43) possess invariant affine lines $x^2 + k = 0$ which could be real, imaginary or could coincide depending on the value of the parameter k (i.e. of the value of N_5). Moreover, since $Z \mid \mathcal{H}$ the line $Z = 0$ could be of the multiplicity two and this is confirmed by the following perturbations:

$$\dot{x} = k + x^2, \quad \dot{y} = (l + ex + fy)(1 + \varepsilon\, y).$$

a) Assume $N_5 > 0$. Then $k < 0$ and since $e \neq 0$ we may consider $k = -1$ and $e = 1$ via the transformation $x \to (-k)^{1/2}x$, $y \to ey$ and $t \to (-k)^{-1/2}t$. So we get the systems

$$\dot{x} = x^2 - 1, \quad \dot{y} = l + x + fy, \tag{4.44}$$

with $f \neq \pm 2$. These systems possess singular points $\big(-1,\ (1-l)/f\big)$ and $\big(1,\ -(1+l)/f\big)$ which tend to infinity when $f \to 0$. We note that for the systems (4.44) we have $D = -f^2x^2y$.

Therefore, for $D \neq 0$ (i.e. $f \neq 0$) we may assume $l = 0$ via the transformation $x \to x$ and $y \to y - l/f$ and taking into account that $Z = 0$ is double we get Config. 4.28.

For $D = 0$ we obtain $f = 0$ and then $l \neq \pm 1$, otherwise we get a degenerate system (4.44). In this case we obtain Config. 4.29.

b) If $N_5 < 0$ then $k > 0$ and since $e \neq 0$ we may consider $k = 1$ and $e = 1$ via the transformation $x \to k^{1/2}x$, $y \to ey$ and $t \to k^{-1/2}t$. So we get the systems

$$\dot{x} = x^2 + 1, \quad \dot{y} = l + x + fy, \tag{4.45}$$

with $l, f \in \mathbb{R}$. These systems have two imaginary invariant lines ($x = \pm i$) and imaginary singular points $\big(-i,\ (i-l)/f\big)$ and $\big(i,\ -(i+l)/f\big)$.

Therefore, for $D = -f^2x^2y \neq 0$ (i.e. $f \neq 0$) we may assume $l = 0$ via a transformation (as above) and taking into account that $Z = 0$ is double we obtain Config. 4.32.

For $D = 0$ we obtain $f = 0$ and then we get Config. 4.33.

c) Assume now that $N_5 = 0$, i.e. $k = 0$. Since $e \neq 0$ we may assume $e = 1$ via the change $y \to ey$. Then we obtain the systems

$$\dot{x} = x^2, \quad \dot{y} = l + x + fy, \tag{4.46}$$

for which the condition $N_2 = 4f^2x \neq 0$ yields $f \neq 0$. Then via the transformation $x_1 = x/f$, $y_1 = y + l/f$ and $t_1 = ft$ we may assume $l = 0$ and $f = 1$. Systems (4.46) possess the invariant line $x = 0$ on which the double point $(0,0)$ is placed. Taking into account that both the lines $x = 0$ and $Z = 0$ are double (this is confirmed by the perturbations $(IV.39_\varepsilon)$ from Table 3) we obtain Config. 4.39.

4.4 Systems with $D_S(C,Z) = 3 \cdot \omega$

In this subsection we shall consider the canonical system $(\mathbf{S}_{IV})$ for which we have: $\theta = 8h^3$.

4.4.1 The case $\theta \neq 0$, $B_3 = 0$

Then $h \neq 0$ and we can assume $c = d = g = 0$ via the affine transformation:

$$x_1 = x - \frac{d}{h}, \quad y_1 = \frac{g}{h}x + y + \frac{ch - dg}{h^2}.$$

Thus we obtain the canonical systems after returning to the same notations for the variables:

$$\dot{x} = k + hxy, \quad \dot{y} = l + ex + fy - x^2 + hy^2. \tag{4.47}$$

We calculate

$$B_3 = 3\,hx^2\left[(ef - 4k)x^2 + 2hlxy - 3hky^2\right], \quad H_7 = -4eh^2,$$

and hence, by $\theta \neq 0$ (i.e. $h \neq 0$) the condition $B_3 = 0$ yields $k = l = fe = 0$. We shall consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

Subcase $H_7 \neq 0$. Then $e \neq 0$ and the condition $B_3 = 0$ yields $k = l = f = 0$. This leads to the systems

$$\dot{x} = hxy, \quad \dot{y} = ex - x^2 + hy^2 \tag{4.48}$$

for which we may assume $e = 1$ and $h \in \{-1, 1\}$ due to the substitution: $x \to ex$, $y \to e|h|^{-1/2}y$ and $t \to e^{-1}|h|^{-1/2}t$. Then for systems (4.48) calculation yields:

$$\begin{aligned}
&\mathcal{E}_1 = 2(X^2 + hY^2 - 2XZ + Z^2)\mathcal{H}, \quad \mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = hX^3,\\
&\mathcal{E}_2 = Y^3\mathcal{H}, \quad \operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8(X - Z)^6.
\end{aligned}$$

So, we obtain $\deg \mathcal{H} = 3$ and since $\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) \not\equiv 0$, according to Lemma 3.8 we cannot have additional nontrivial factors of $\mathcal{E}_1$ and $\mathcal{E}_2$.

The invariant line $x = 0$ is of the multiplicity 3 and this is confirmed by the perturbations $(IV.45_\varepsilon)$ from Table 3. Taking into account that systems (4.48) possess the singular points $(0,0)$ (triple) and $(1,0)$ (simple), and the last point is not located on the invariant line $x = 0$, we obtain Config. 4.45.

Subcase $H_7 = 0$. In this case the condition $B_3 = 0$ yields $k = l = e = 0$ and we obtain the systems

$$\dot{x} = hxy, \quad \dot{y} = fy - x^2 + hy^2. \tag{4.49}$$

Calculations yield:

$$\mathcal{E}_1 = 2(X^2 + hY^2 - fYZ)\mathcal{H}, \quad \mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = hX^3,$$
$$\mathcal{E}_2 = hY^3\mathcal{H}, \quad \mathrm{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8h^4X^6.$$

So, deg $\mathcal{H} = 3$ and from $h \neq 0$ we have $\mathrm{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) \not\equiv 0$, i.e. we cannot have an additional nontrivial factor of $\mathcal{E}_1$ and $\mathcal{E}_2$. The invariant line $x = 0$ is of the multiplicity three and this is confirmed by the perturbations $(IV.41_\varepsilon)$ (for $f \neq 0$) and $(IV.42_\varepsilon)$ (for $f = 0$) from Table 3. We observe that systems (4.49) possess the singular points $(0, 0)$ (triple) and $(0, -f/h)$ (simple), and for these systems $D = -f^2x^3$.

Therefore, if $D \neq 0$ then $f \neq 0$ and we may assume $f = 1$, $h \in \{-1, 1\}$ via the substitution $x \to f|h|^{-1/2}x$, $y \to f|h|^{-1}y$ and $t \to f^{-1}t$. Since in this case the singular points above are distinct we obtain Config. 4.41.

Assume $D = 0$, i.e. $f = 0$. Then systems (4.49) possess one point $(0, 0)$ of multiplicity four and we get Config. 4.42. Note that in this case we may assume $h \in \{-1, 1\}$ via the substitution $x \to |h|^{-1/2}x$, $y \to |h|^{-1}y$.

4.4.2 The case $\theta = 0 = B_2$

The condition $\theta = 0$ yields $h = 0$ and then

$$B_2 = -648d^2\big[\big((d + cg - fg)^2 + g^2(2f^2 - cf - gk)\big)x^2 + 4dg(d + cg - 3fg)xy - 6d^2g^2y^2\big]x^2.$$

The condition $B_2 = 0$ yields $d = 0$ and calculations yield:

$$N = g^2x^2, \qquad B_3 = 3g(cf - f^2 - gk)x^4 \tag{4.50}$$

and we shall consider two subcases: $N \neq 0$ and $N = 0$.

$\boxed{\textbf{Subcase } N \neq 0.}$ Then $g \neq 0$ and we may assume $g = 1$ and $e = f = 0$ via the transformation

$$x_1 = x + f/g, \quad y_1 = gy + (2f + eg)/g, \quad t_1 = g\,t.$$

So keeping the previous notations we obtain the systems

$$\dot{x} = k + cx + x^2, \quad \dot{y} = l - x^2 + xy \tag{4.51}$$

for which we calculate:

$$\mathcal{E}_1 = 2\big[X^3 + cX^2Z + (2k + l)XZ^2 - kYZ^2 + clZ^3\big]\mathcal{H},$$
$$\mathcal{E}_2 = (X^2 + cXZ + kZ^2)^2\mathcal{H}, \quad \mathcal{H} = (X^2 + cXZ + kZ^2),$$
$$\mathrm{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 16k^2Z^8\left[kY^2 + c(k - l)YZ + (k^2 - c^2l + 2kl + l^2)Z^2\right]^2.$$

Thus, we have deg $\mathcal{H} = 2$ and according to Lemma 3.8 to have an additional factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ we must have $\mathrm{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 0$ in $\mathbb{R}[Y, Z]$. Therefore, we obtain the condition $k = 0$ which is equivalent to $B_3 = 0$ since $B_3 = -3ky^4$. So, we get the systems

$$\dot{x} = cx + x^2, \quad \dot{y} = l - x^2 + xy \tag{4.52}$$

for which we calculate again:

$$\mathcal{E}_1 = 2(X^2 + lZ^2)\mathcal{H}, \quad \mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = X(X + cZ)^2,$$
$$\mathcal{E}_2 = X^2(X + cZ)\mathcal{H}, \quad \mathrm{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 8l^2(c^2 + l)Z^6.$$

The condition $l=0$ yields degenerate systems (4.52). Hence, in view of Lemma 3.8 in order to have $\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})\not\equiv 0$ the condition $c^2+l\neq 0$ has to be satisfied. On the other hand for systems (4.52) we have $N_6=8(c^2+l)x^3$. So, the condition $c^2+l\neq 0$ is equivalent to $N_6\neq 0$. We observe that systems (4.52) possess invariant lines: $x=0$ and $x+c$=0. Moreover, the last line is of the multiplicity two, and for $c=0$ the line $x=0$ is of the multiplicity three. This is confirmed by the perturbations ($IV.31_\varepsilon$) (for $c\neq 0$) and ($IV.44_\varepsilon$) (for $c=0$) from Table 3.

On the other hand for systems (4.52) we have $H_{11}=48c^2x^4$ and hence the condition $c=0$ is equivalent to $H_{11}=0$. So, for $H_{11}\neq 0$ we obtain $c\neq 0$ and we may assume $c=1$ via Remark 4.3 ($\gamma=c,\ s=1$). This leads to Config. 4.31.

Assuming $c=0$ since $l\neq 0$ we may consider $l\in\{-1,1\}$ via Remark 4.3 ($\gamma=|l|,\ s=1/2$) and we obtain Config. 4.44.

$\boxed{\textbf{Subcase } N=0\textbf{.}}$ Then $g=0$ and from (4.50) we obtain $B_3=0$. We may assume $e=0$ via the translation $x\to x+e/2$ and $y\to y$ and therefore we obtain the systems

$$\dot{x}=k+cx,\quad \dot{y}=l+fy-x^2 \tag{4.53}$$

for which calculations yield:

$$\begin{aligned}&\mathcal{E}_1=2\big[(c+f)X^2+2kXZ+f(c-f)YZ+l(c-f)Z^2\big]\mathcal{H},\\ &\mathcal{E}_2=Z(cX+kZ)^2\mathcal{H},\quad \mathcal{H}=\gcd(\mathcal{E}_1,\mathcal{E}_2)=Z^2(cX+kZ).\end{aligned}$$

Thus, $\deg\mathcal{H}=3$ and, since the polynomial $\mathcal{E}_2$ does not depend on Y, according to Lemma 3.8 we could have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ only if at least one of the following conditions holds:

$$\begin{aligned}&\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=4(c-f)^2Z^4\left[c^2fY+(c^2l-k^2)Z\right]^2\equiv 0,\\ &\operatorname{Res}_Z(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=8(c-f)^2(c+f)X^4\left[(c^2l-k^2)X-cfkY\right]^2\equiv 0,\end{aligned}$$

which amount to at least one of the following conditions:

$$(i)\ (c-f)(c+f)=0;\quad (ii)\ cf=c^2l-k^2=0;\quad (iii)\ cfk=c^2l-k^2=0.$$

We observe, that the conditions (ii) yield either $c=k=0$ or $f=c^2l-k^2=0$ and both these cases lead to degenerate systems (4.53). If $k\neq 0$ the conditions (iii) are equivalent to (ii), and for $k=0$ we obtain $l=0$. However the conditions $k=l=0$ do not imply the existence of an additional common factor for the polynomials $\mathcal{E}_i/\mathcal{H}$ $(i=1,2)$ unless $c+f=0$ which falls in the case (i).

Thus, we do not have an additional common factor for the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ if $c^2-f^2\neq 0$.

We observe that the systems (4.53) possess invariant affine lines $cx+k=0$ only for $c\neq 0$. On the other hand, for these systems we have

$$N_3=3(c-f)x^3,\quad D_1=c+f,\quad N_6=8c(c-f)x^3.$$

Hence, the condition $c^2-f^2\neq 0$ is equivalent to $D_1N_3\neq 0$ and for $N_3\neq 0$ the condition $c=0$ is equivalent to $N_6=0$.

1) Assume firstly $N_6\neq 0$, i.e. $c\neq 0$. Then we may consider $c=1$ via Remark 4.3 ($\gamma=c,\ s=1$). In this case the systems (4.53) possess the invariant straight lines $x+k=0$ and the singular point $\big(-k,(k^2-l)/f\big)$ which is finite for $f\neq 0$ and it tends to infinity when f tends to zero. Since for systems (4.53) $D=-f^2x^3$ we conclude that the condition $f\neq 0$ is captured by the T-comitant D.

Assuming $D \neq 0$ we obtain $f \neq 0$ and then we may assume $l = 0$ due to a translation. Thus we obtain the systems

$$\dot{x} = k + x, \quad \dot{y} = fy - x^2 \tag{4.54}$$

with $f(f^2-1) \neq 0$. Moreover we may assume $k \in \{0, 1\}$ via the rescaling $x \to kx$ and $y \to k^2 y$ (for $k \neq 0$). Taking into account that the line $Z = 0$ is triple (and this is confirmed by the perturbations $(IV.37_\varepsilon)$ from Table 3) we get Config. 4.37.

If $D = 0$ we have $f = 0$ and this leads to the systems

$$\dot{x} = k + x, \quad \dot{y} = l - x^2 \tag{4.55}$$

with $l - k^2 \neq 0$ and as above we may assume $k \in \{0, 1\}$. Taking into account that these systems do not have finite singular points and the line $Z = 0$ is triple (this is confirmed by the perturbations $(IV.38_\varepsilon)$ from Table 3) we get Config. 4.38.

2) Assume now $N_6 = 0$, i.e. $c = 0$. Then the condition $D_1 N_3 \neq 0$ yields $f \neq 0$ and since $k \neq 0$ for systems (4.53) we may assume $k = f = 1$ and $l = 0$ via the transformation

$$x_1 = fk^{-1}x, \quad y_1 = f^3k^{-2}y + lf^2k^{-2}, \quad t_1 = ft.$$

Hence we obtain the system

$$\dot{x} = 1, \quad \dot{y} = y - x^2 \tag{4.56}$$

for which the line $Z = 0$ is of the multiplicity four as it is confirmed by the perturbations $(IV.46_\varepsilon)$ from Table 3. Thus in this case we get Config. 4.46.

All the cases in Theorem 4.1 are thus examined. To finish the proof of the Theorem 4.1 it remains to show that the conditions occurring in the middle column of Table 2 are affinely invariant. This follows from the proof of Lemma 4.3. ∎

Lemma 4.3. *The polynomials which are used in Theorem 4.1 have the properties indicated in the Table 4. In the last column are indicated the algebraic sets on which the GL-comitants on the left are CT-comitants.*

Proof: I. Cases 1–17. Assume that $\boldsymbol{a} \in \mathbb{R}^{12}$ corresponds to an arbitrarily given system (2.1) and assume $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ corresponds to a system in the orbit of the given system (2.1) under the action of the translation group, i.e. if $\tau: \ x = \tilde{x} + x_0, \ y = \tilde{y} + y_0$ then

$$\tilde{\boldsymbol{a}}: \quad \begin{cases} \dot{\tilde{x}} = P(\boldsymbol{a}, x_0, y_0) + P_x(\boldsymbol{a}, x_0, y_0)\tilde{x} + P_y(\boldsymbol{a}, x_0, y_0)\tilde{y} + p_2(\boldsymbol{a}, \tilde{x}, \tilde{y}), \\ \dot{\tilde{y}} = Q(\boldsymbol{a}, x_0, y_0) + Q_x(\boldsymbol{a}, x_0, y_0)\tilde{x} + Q_y(\boldsymbol{a}, x_0, y_0)\tilde{y} + q_2(\boldsymbol{a}, \tilde{x}, \tilde{y}). \end{cases}$$

Then for every $\boldsymbol{a} \in \mathbb{R}^{12}$ and $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2$ calculations yield:

$$\begin{aligned} &U(\tilde{\boldsymbol{a}}) = U(\boldsymbol{a}) \quad \text{for each} \quad U \in \{\eta, \mu, \theta, B_1, H_1, H_4, H_5, H_7, H_8, H_9, H_{10}\}, \\ &W(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = W(\boldsymbol{a}, \tilde{x}, \tilde{y}) \quad \text{for each} \quad W \in \{C_2, K, H, M, N, D, B_2, B_3, H_2, H_3, H_6, H_{11}\}. \end{aligned}$$

Hence according to the definition of T-comitants (see [24]) we conclude that the GL- comitants indicated in the lines 1–17 of Table 4 are T-comitants for systems (2.1).

II. Cases 18–24. **1)** We consider firstly the GL-comitants $N_1(a, x, y)$, $N_2(a, x, y)$ and $N_5(a, x, y)$ and we shall prove that N_2 and N_5 (respectively N_1) are CT-comitants modulo $\langle \eta, N, K, B_3 \rangle$ (respectively modulo $\langle \eta, N, K \rangle$). We shall examine the two subcases: $M \neq 0$ and $M = 0$.

a) For $\eta = 0$ and $M \neq 0$ we are in the class of the systems $(\mathbf{S}_{III})$, for which the conditions $N = (g^2-1)x^2 + 2h(g-1)xy + h^2y^2 = 0$ and $K = 2g(g-1)x^2 + 4ghxy + h^2y^2 = 0$ yield

Table 4

Case	GL-comitants	Degree in a	Degree in x and y	Weight	Algebraic subset $V(*)$
1	$\eta(a)$, $\mu(a)$, $\theta(a)$	4	0	2	$V(0)$
2	$C_2(a,x,y)$	1	3	-1	$V(0)$
3	$H(a,x,y),\ K(a,x,y)$	2	2	0	$V(0)$
4	$M(a,x,y),\ N(a,x,y)$	2	2	0	$V(0)$
5	$D(a,x,y)$	3	3	-1	$V(0)$
6	$B_1(a)$	12	0	3	$V(0)$
7	$B_2(a,x,y)$	8	4	0	$V(0)$
8	$B_3(a,x,y)$	4	4	-1	$V(0)$
9	$H_1(a)$	6	0	2	$V(0)$
10	$H_2(a,x,y)$	3	2	0	$V(0)$
11	$H_3(a,x,y)$	4	2	0	$V(0)$
12	$H_4(a),\ H_{10}(a)$	6	0	2	$V(0)$
13	$H_5(a),\ H_8(a)$	8	0	2	$V(0)$
14	$H_6(a,x,y)$	8	6	0	$V(0)$
15	$H_7(a)$	3	0	1	$V(0)$
16	$H_9(a))$	12	0	2	$V(0)$
17	$H_{11}(a,x,y))$	6	4	0	$V(0)$
18	$N_1(a,x,y)$	3	4	-1	$V(\eta,N,K)$
19	$N_2(a,x,y)$	3	1	0	$V(\eta,N,K,B_3)$
20	$N_3(a,x,y)$	2	3	-1	$V(M,N)$
21	$N_4(a,x,y)$	2	2	-1	$V(M,N,N_3)$
22	$N_5(a,x,y)$	4	2	0	$V(\eta,N,K,B_3)$
23	$N_6(a,x,y)$	3	3	-1	$V(M,\theta,B_3)$
24	$D_1(a)$	1	0	0	$V(M,N)$

$h = g - 1 = 0$. Hence applying the additional translation $x \to x - c/2$, $y \to y$ we obtain the systems

$$\dot{x} = k + dy + x^2, \qquad \dot{y} = l + ex + fy. \tag{4.57}$$

On the other hand for any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (4.57) calculations yield:

$$N_1(\tilde{\boldsymbol{a}},\tilde{x},\tilde{y}) = 8\tilde{x}^2(e\tilde{x}^2 - 2d\tilde{y}^2), \quad N_2(\tilde{\boldsymbol{a}},\tilde{x},\tilde{y}) = 4(f^2+4k)\tilde{x} + 4df\tilde{y} + 8d(x_0\tilde{y} + 2y_0\tilde{x}),$$
$$N_5(\tilde{\boldsymbol{a}},\tilde{x},\tilde{y}) = -16(4k\tilde{x}^2 - d^2\tilde{y}^2) + 64d\tilde{x}(x_0\tilde{y} - y_0\tilde{x}), \quad B_3(\tilde{\boldsymbol{a}},\tilde{x},\tilde{y}) = 6d\tilde{x}\tilde{y}^2(f\tilde{x} - d\tilde{y}).$$

Hence the polynomial N_1 does not depend on the vector defining the translations and for $B_3 = 0$ the same occurs for the polynomials N_2 and N_5. Therefore we conclude that for $M \neq 0$ the polynomial N_1 is a CT-comitant modulo $\langle \eta, N, K, \rangle$, whereas the polynomials N_2 and N_5 are CT-comitants modulo $\langle \eta, N, K, B_3 \rangle$.

b) Assume now that $M = 0$. Then we are in the class of the systems $(\mathbf{S}_{IV})$, for which the condition $N = (g^2 - 2h)x^2 + 2ghxy + h^2y^2 = 0$ yields $h = g = 0$ and then $K = 0$. Then applying

the additional translation $x \to x + e/2$, $y \to y$, we obtain the systems

$$\dot{x} = k + cx + dy, \qquad \dot{y} = l + fy - x^2. \tag{4.58}$$

For any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (4.58) calculations yield:

$$N_1(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = -24d\tilde{x}^4, \quad N_2(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 12d(c+f)\tilde{x}, \quad N_5(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 0 = B_3(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}).$$

Since the condition $M = 0$ implies $\eta = 0$, considering the case **1 a)** above, we conclude that independently of either $M \neq 0$ or $M = 0$ the GL-comitant N_1 is a CT-comitant modulo $\langle \eta, N, K \rangle$ and N_2 and N_5 are CT-comitants modulo $\langle \eta, N, K, B_3 \rangle$.

2) Let us now consider the GL-comitants $N_3(a, x, y)$, $N_4(a, x, y)$, $N_6(a, x, y)$ and $D_1(a)$. According to Table 4 we only need to examine the class of the systems $(\mathbf{S}_{IV})$ and we shall consider the two subcases: $N = 0$ and $N \neq 0$, $\theta = 0$.

a) If for a system $(\mathbf{S}_{IV})$ the condition $N = 0$ is fulfilled then it was shown above that this system can be brought via a translation to the form (4.58). For any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (4.58) calculations yield:

$$\begin{aligned}
&N_3(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 3(c-f)\tilde{x}^3 + 2d\tilde{x}^2\tilde{y}, \quad B_3(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 6d\tilde{x}^3(f\tilde{x} - d\tilde{y}),\\
&N_4(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 12k\tilde{x}^2 + 3(f^2 - c^2)\tilde{x}\tilde{y} - 3d(c+f)\tilde{y}^2 + 6\tilde{x}^2[(c-f)x_0 + 2dy_0],\\
&N_6(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 8c(c-f)\tilde{x}^3 + 16df\tilde{x}^2\tilde{y} - 8d^2\tilde{x}\tilde{y}^2 - 48dx_0\tilde{x}^3, \quad D_1(\tilde{\boldsymbol{a}}) = c + f.
\end{aligned}$$

These relations show us that: *(i)* the GL-comitants N_3 and D_1 are CT-comitant modulo $\langle M, N \rangle$; *(ii)* the GL-comitant N_4 is a CT-comitant modulo $\langle M, N, N_3 \rangle$; *(iii)* the GL-comitant N_6 is a CT-comitant modulo $\langle M, N, B_3 \rangle$.

b) Assume that for the systems $(\mathbf{S}_{IV})$ the conditions $\theta = 8h^3 = 0$ and $N = (g^2 - 2h)x^2 + 2ghxy + h^2y^2 \neq 0$ are fulfilled. Then $h = 0$, $g \neq 0$ and we may assume $g = 1$ and $e = f = 0$ via the transformation $x_1 = x + f/g$, $y_1 = gy + (2f + eg)/g$, $t_1 = g\,t$. Then we obtain the systems

$$\dot{x} = k + cx + dy + x^2, \quad \dot{y} = l - x^2 + xy$$

for which calculation yields: $B_3 = -3x^2\big[kx^2 + 2d(c+d)xy + 3d^2y^2$. Therefore the condition $B_3 = 0$ yields $k = d = 0$ and we obtain the following family of systems which is characterized by the conditions $M = \theta = B_3 = 0$ and $N \neq 0$:

$$\dot{x} = cx + x^2, \quad \dot{y} = l - x^2 + xy. \tag{4.59}$$

For any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (4.59) we have $N_6(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 8(l + 3f^2)\tilde{x}^3$. Since the condition $N = 0$ implies $\theta = 0$, considering the case **2 a)** above, we conclude in this case that independently of either $N \neq 0$ or $N = 0$, the GL-comitant N_6 is a CT-comitant modulo $\langle M, \theta, B_3 \rangle$.

The Table 4 show us that all the conditions indicated in the middle column of Table 2 are affinely invariant. Indeed, the CT-comitants N_i, $i = 1, \ldots, ..., 6$ and D_1 are used in Table 2 only for the varieties indicated in the last column of Table 4. This complete the proof of the Theorem 4.1. ∎

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